Mathematics, Pusan National University

MATRIX ANALYSIS

2.1. Unitrary matrices and the QR Factorization

Taehyeong Kim
th_kim@pusan.ac.kr

September 24, 2020

Content

Definition

A list of vectors $x_1, \ldots, x_k \in \mathbf{C}^n$ is **orthogonal** if $x_i^* x_j = 0$ for all $i \neq j, i, j \in \{1, \ldots, k\}$. If, in addition, $x_i^* x_i = 1$ for all $i = 1, \ldots, k$, then the list is **orthonormal**. Ir is often convenient to say that "$x_1, \ldots, x_k$ are orthogonal (respectively, orthonormal)" instead of the nore formal statement "the list of vector $v_1 \ldots, v_k$ is orthogonal (orthonormal, respectively)."

Theorem

Every orthonormal list of vector in $\mathbf{C}^n$ is linearly independent.

Proof.

Suppose that $x_1, \ldots, x_k$ is an orthonomal set, and suppose that $0 = \alpha_1 x_1 + \cdots + \alpha_k x_k$. Then

$$\begin{aligned} 0 &= (\alpha_1 x_1 + \cdots + \alpha_k x_k)^* (\alpha_1 x_1 + \cdots + \alpha_k x_k) \\ &= \sum_{i,j} \bar{\alpha}_i \alpha_j x_i^* x_j \\ &= \sum_{i=1}^{k} |\alpha_i|^2 \end{aligned}$$

because the vectors x_i are orthogonal and normalized. Thus, all $\alpha_i = 0$ and hence $\{x_i, \ldots, x_k\}$ is linearly independent set. □

Unitrary matrices and the QR factorization

Definition

A matrix $U \in M_n$ is unitrary if $U^*U = I$. A matrix $U \in M_n(R)$ is real orthogoanl if $U^TU = I$.

Example

$U \in M_n$ and $V \in M_m$ are unitrary if and only if $U \oplus V \in M_{m+n}$ is unitrary.

Theorem

If $U \in M_n$, the following are equivalent:

- U *is unitary*
- U *is nonsingular and* $U^* = U^{-1}$
- $UU^* = I$
- U^* *is unitary*
- *The columns of* U *are orthonormal*
- *The rows of* U *are orthonormal*
- *For all* $x \in \mathbf{C}^n$, $\|x\|_2 = \|Ux\|_2$

Definition

A linear transformation $T : \mathbf{C}^n \to \mathbf{C}^m$ is called a **Euclidean isometry** if $\|x\|_2 = \|Tx\|_2$ for all $x \in \mathbf{C}^n$. A square complex matrix $U \in M_n$ is a **Euclidean isometry** if and only if it is unitrary. In chapter 5, there are other kindes of isometries.

If U, V are unitrary, then UV is also unitrary.
$(\because (UV)(UV)^* = U(VV^*)U^* = UU^* = I)$
The set of unitrary matrices in M_n forms a group.

Lemma

Let $U_1, U_2, \ldots \in M_n$ are given infinite sequence of unitrary matrices. There exists an infinite subsequence $U_{k_1}, U_{k_2}, \ldots, 1 \leq k_1 < k_2 < \cdots$, such that all of the entries of U_{k_i} converge to the entries of a unitrary matrix as $i \to \infty$.

Theorem

Let $A \in M_n$ be nonsingular. Then A^{-1} is similar to A^* if and only if there is a nonsingular $B \in M_n$ such that $A = B^{-1}B$.

Proof.

$(\Leftarrow)$

$$
\begin{aligned}
A = B^{-1}B \Rightarrow A^{-1} &= (B^*)^{-1}B \\
\Rightarrow B^*A^{-1}(B^*)^{-1} &= B(B^*)^{-1} = ((B^*)^{-1}B)^* = A^*
\end{aligned}
$$

Theorem

Let $A \in M_n$ be nonsingular. Then A^{-1} is similar to A^* if and only if there is a nonsingular $B \in M_n$ such that $A = B^{-1}B$.

Proof (Cont.)

$(\Rightarrow)$
A^{-1} is similar to A^*. Then there is a nonsingular $S \in M_n$ such that $SA^{-1}S^{-1} = A^* \Rightarrow S = A^*SA$.

$$S_\theta = A^*S_\theta A,\ S_\theta^* = A^*S_\theta^* A \text{ where } S_\theta = e^{i\theta}S$$

$$H_\theta = A^*H_\theta A \text{ where } H_\theta = S_\theta + S_\theta^*$$

If H_θ were singular, there would be a nonzero $x \in \mathbf{C}^n$ such that $0 = H_\theta x = S_\theta x + S_\theta^* x$

Theorem

Let $A \in M_n$ be nonsingular. Then A^{-1} is similar to A^* if and only if there is a nonsingular $B \in M_n$ such that $A = B^{-1}B$.

Proof (Cont.)

So $-x = S_\theta^{-1} S_\theta^* x = e^{2i\theta} S^{-1} S^* x = e^{2i\theta} x$.
Choose $\theta_0 \in [0, 2\pi)$ such that $-e^{2i\theta_0}$ is not an eigenvalue of $S^{-1}S^*$. The resulting Hermitian matrix $H = H_{\theta_0}$ is nonsingular and has the property that $H = A^* H A$. □

Lemma

Let a unitrary $U \in M_n$ be partitioned as $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, in which $U_{11} \in M_k$. Then rankU_{12} =rankU_{21} and rankU_{22} =rank$U_{11} + n - 2k$. In particular, $U12 = 0$ if and only if $U_{21} = 0$, in which case U_{11} and U_{22} are unitary.

Example

Householder matrices Let $\omega \in \mathbf{C}^n$ be nonzero vector. The **Householder matrix** $U_\omega \in M_n$ is defined by $U_\omega = I - 2(\omega^* \omega)^{-1} \omega \omega^*$. If ω is a unit vector $U_\omega = I - 2\omega\omega^*$.

Unitrary matrices and the QR factorization

Theorem

Let $x, y \in \mathbf{C}^n$ be given and suppose that $\|x\|_2 = \|y\|_2 > 0$. If $y = e^{i\theta} x$ for some real θ, let $U(y,x) = e^{i\theta} I_n$; otherwise, let $\phi \in [0, 2\pi)$ be such that $x^* y = e^{i\phi}|x^* y|$; let $\omega = e^{i\phi} x - y$; and let $U(y,x) = e^{i\phi} U_\omega$ where $U_\omega = I - 2(\omega^*\omega)^{-1}\omega\omega^*$ is a Householder matrix. Then $U(y,x)$ is unitary and essentially Hermitian, $U(y,x)x = y$, and $U(y,x)z \perp y$ whenever $z \perp x$. If x and y are real, then $U(y,x)$ is real orthogonal: $U(y,x) = I$ if $y = x$, and $U(y,x)$ is the real Householder matrix U_{x-y} otherwise.

Unitary matrices and the QR factorization

Theorem

Let $A \in M_{n,m}$ be given.

(a) *If $n \geq m$, there is a $Q \in M_{n,m}$ with orthonormal columns and an upper triangular $R \in M_m$ with nonnegative main diagonal entries such that $A = QR$.*

(b) *If rank$A = m$, then the factors Q and R in (a) are uniquely determined and the main diagonal entries of R are all positive.*

(c) *If $m = n$, then the factor Q in (a) is unitary.*

(d) *There is a unitary $Q \in M_n$ and an upper triangular $R \in M_n, m$ with nonnegative diagonal entries such that $A = QR$.*

(e) *If A is real, then the factors Q and R in (a), (b), (c), and (d) may be taken to be real.*

Unitary matrices and the QR factorization

Theorem

If $X = [x_1, \ldots, x_k] \in M_{n,k}$ and $Y = [y_1, \ldots, y_k] \in M_{n,k}$ have orthonormal columns, then there is a unitary $U \in M_n$ such that $Y = UX$. If X and Y are real, then U may be taken to be real.

Thank you!