# Zhang Neural Network and Generalized Linear Matrix Equation

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# Content

# Concept of Zhang Dynamics & Zhang Function

Consider the time-varying reciprocal problem in the following form:

$$f(x(t),t) = a(t)x(t) - 1 = 0 \in \mathbb{R}, t \in [0, -\infty) \quad (1)$$

where $a(t) \neq 0 \in \mathbb{R}$ denotes a smoothly time-varying scalar with $\dot{a}(t) \in \mathbb{R}$ denoting the time derivative of $a(t)$.
**aim :** Finding the $x(t) \in \mathbb{R}$ to make (1) hold true at any time $t \in [0, -\infty)$.
And denote $x^*(t)$ as the theoretical time-varying reciprocal of $a(t)$, i.e., mathematically, $x^*(t) = 1/a(t)$ in (1).

## Remark

This $x^*(t)$ is given symbolically for better understanding and solution comparison, whose the computation of $1/a(t)$ at every single time instant $t$ is less practical in real-life applications. When we compute $1/a(t)$ at a time instant $t$, as the computation consumes time $\Delta t$ inevitably, the value of $a(t)$ is changing during the computation procedure. This is the so-called **lagging error phenomenon**.

# Concept of Zhang Dynamics & Zhang Function

Zhang dynamics (ZD) has been formally proposed by Zhang et al. for various time-varying problems solving.

## Concept of Zhang dynamics

Zhang dynamics(ZD) is a special type of neural dynamics that has been formally proposed by Zhang et al. for various time-varying problems solving.

According to Zhang et al.'s neural-dynamics design method, the ZD is designed based on an indefinite Zhang function (ZF) as the error-monitoring function.

## Concept of Zhang function

1. indefinite (i.e., can be positive, zero, or negative, in addition to being bounded, unbounded, or even lower unbounded)
2. can be matrix or vector valued
3. can be real or complex valued to monitor and control the process of time-varying problems solving fully.

# Concept of Zhang Dynamics & Zhang Function

To lay a basis for further discussion, the design procedure for a ZD model is presented as follows.

1. Define an indefinite ZF as the error-monitoring function to monitor the process of time-varying reciprocal finding.
2. To force $e(t)$ globally and exponentially converge to zero, we choose its time derivative $\dot{e}(t)$ via the following ZD design formula,

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma e(t), \tag{2}$$

   where design parameter $\gamma > 0 \in \mathbb{R}$.
3. By expanding the ZD design formula (2), the dynamic equation of a ZD model is thus established for time-varying reciprocal finding.

# Concept of Zhang Dynamics & Zhang Function

**Theorem 1.1**

As for the ZD design formula (2) which is also a dynamic system, starting from an initial error $e(0) \in \mathbb{R}$, the error function $e(t) \in \mathbb{R}$ globally and exponentially converges to zero with rate $\gamma$.

**Proof.**

For (2), by calculus, we obtain its analytical solution as $e(t) = e(0)exp(-\gamma t)$. Based on the definition of global and exponential convergence, we can draw the conclusion that, starting from any $e(0)$, $e(t)$ globally and exponentially converges to zero with rate $\gamma$, as time $t$ tends to infinity. □

# Concept of Zhang Dynamics & Zhang Function

For real-time solution of time-varying reciprocal problem (1), we define the following four different ZFs:

$$e(t) = x(t) - \frac{1}{a(t)}, \tag{3}$$

$$e(t) = a(t) - \frac{1}{x(t)}, \tag{4}$$

$$e(t) = a(t)x(t) - 1, \tag{5}$$

$$e(t) = \frac{1}{a(t)x(t)} - 1. \tag{6}$$

# Concept of Zhang Dynamics & Zhang Function

## Example of ZD model

Let us consider the ZD design formula (2) and ZF (3). Then, we have

$$\dot{x}(t) + \frac{1}{a^2(t)}\dot{a}(t) = -\gamma\left(x(t) - \frac{1}{a(t)}\right),$$

which is rewritten as

$$a^2(t)\dot{x}(t) + = -\dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right). \quad (7)$$

Thus, we obtain ZD model (7) for time-varying reciprocal finding.

# Concept of Zhang Dynamics & Zhang Function

Similarly, we obtain ZD models using ZFs equations (4)–(6), respectively.

| ZF | ZD model |
|---|---|
| (3) | $a^2(t)\dot{x}(t) = -\dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right)$ |
| (4) | $\dot{x}(t) = -\dot{a}(t)x^2(t) - \gamma\left(a(t)x^2(t) - x(t)\right)$ |
| (5) | $a(t)\dot{x}(t) = -\dot{a}(t)x(t) - \gamma(a(t)x(t) - 1)$ |
| (6) | $a(t)\dot{x}(t) = -\dot{a}(t)x(t) + \gamma\left(a(t)x(t) - a^2(t)x^2(t)\right)$ |

Table 1: Different ZFs resulting in different ZD models for time-varying reciprocal finding

# Concept of Zhang Dynamics & Zhang Function

We show following proposition which show the convergence properties of the proposed ZD model (7) for time-varying reciprocal finding.

## Proposition

Consider a smoothly time-varying scalar $a(t) \neq 0 \in \mathbb{R}$ involved in time-varying reciprocal problem (1). Starting from randomly-generated initial state $x(0) \neq 0 \in \mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (7) derived from ZF (3) exponentially converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$].

## Concept of Zhang Dynamics & Zhang Function

**Proof.**

We use the well-known Lyapunov method to prove the exponential convergence of ZD model (7)
First, starting with ZF (3), we define a Lyapunov candidate

$$V(x(t),t)=\frac{1}{2}\left(x(t)-\frac{1}{a(t)}\right)^2\geqslant 0,$$

where $V(x(t),t)=0$ for any $x(t)=a^{-1}(t)$, and $V(x(t),t)>0$ for any $x(t)\neq a^{-1}(t)$. Then, we derive its time derivative as

$$\begin{aligned}\dot{V}(x(t),t)=\frac{\mathrm{d}V(x(t),t)}{\mathrm{d}t}&=\left(x(t)-\frac{1}{a(t)}\right)\left(\dot{x}(t)+\frac{1}{a^2(t)}\dot{a}(t)\right)\\&=-\gamma\left(x(t)-\frac{1}{a(t)}\right)^2=-2\gamma V(x(t),t)\end{aligned} \tag{8}$$

# Concept of Zhang Dynamics & Zhang Function

**Proof.**

Since $V(x(t),t) \geq 0$, then $\dot{V}(x(t),t) = -2\gamma V(x(t),t) \leq 0$, which guarantees the (final) negative-definiteness of $\dot{V}(x(t),t)$.
Furthermore, from $\dot{V}(x(t),t) = -2\gamma V(x(t),t)$, we have

$$V(x(t),t) = V(x(0),0)exp(-2\gamma t).$$

That is,

$$\frac{1}{2}\left(x(t) - \frac{1}{a(t)}\right)^2 = \frac{1}{2}\left(x(0) - \frac{1}{a(0)}\right)^2 \exp(-2\gamma t).$$

Thus, we have

$$\left|x(t) - \frac{1}{a(t)}\right| = \left|x(0) - \frac{1}{a(0)}\right| \exp(-\gamma t),$$

where symbol $|\cdot|$ denotes the absolute value of a scalar.

# Concept of Zhang Dynamics & Zhang Function

**Proof.**

With $\alpha = |x(0) - 1/a(0)|$, the above equation is further rewritten as

$$\left| x(t) - \frac{1}{a(t)} \right| = \alpha \exp(-\gamma t),$$

which means that $x(t)$ exponentially converges to $a^{-1}(t)$ with the convergence rate $\gamma > 0$. That is, starting from randomly-generated initial state $x(0) \neq 0 \in \mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (7) exponentially converges to the theoretical time-varying reciprocal $x^*(t) = a^{-1}(t)$ of $a(t)$ involved in time-varying Eq. (1). □

# Concept of Zhang Dynamics & Zhang Function

For ZD model (7),

$$\dot{x}(t) = \left(1 - a^2(t)\right)\dot{x}(t) - \dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right).$$

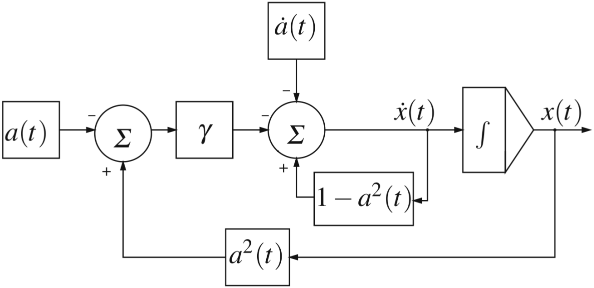


Figure 1: Block diagrams of ZD models (7) for time-varying reciprocal finding

## Time-Varying Matrix Equation

In recent years, the problem of solving linear matrix equations, e.g., Sylvester equation, Lyapunov equation, and Stein's equation, has been encountered in various science and engineering fields.

- Sylvester equation

$$AX + XB = C$$

- Lyapunov equation

$$AXA^H - X + Q = 0 \quad \text{(discrete Lyapunov equation)}$$
$$AX + XA^H + Q = 0 \quad \text{(continuous Lyapunov equation)}$$

- Stein's equation

$$AXB - X + Q = 0$$

- Riccati Equation

$$XQX + XA + A^H X - C = 0$$

# Time-Varying Matrix Equation

We will prove

$$A(t)X(t) - I = 0 \in \mathbb{R}^{n\times n} \tag{9}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is the smoothly time-varying nonsingular coefficient matrix. Note that $A(t)$ together with its time derivative $\dot{A}(t) \in \mathbb{R}^{n\times n}$ is assumed to be known or measurable.

Generally, if the time-varying matrix $A(t) \in \mathbb{R}^{m\times n}$ is of full-rank, i.e., $\mathrm{rank}(A) = \min\{m, n\}$ at any time instant $t \in [0, +\infty)$, then the unique time-varying pseudoinverse/inverse $A^{+}(t)$ for matrix $A(t)$

$$A^{+}(t) = \begin{cases} \left(A^{\mathrm{T}}(t)A(t)\right)^{-1} A^{\mathrm{T}}(t), & \text{if } m > n \\ A^{-1}(t), & \text{if } m = n \\ A^{\mathrm{T}}(t)\left(A(t)A^{\mathrm{T}}(t)\right)^{-1}, & \text{if } m < n \end{cases} \tag{10}$$

# Time-Varying Matrix Equation

ZD design formula (2) is further generalized as follows

$$\dot{E}(t) = \frac{dE(t)}{dt} = -\gamma E(t), \tag{11}$$

where design parameter $\gamma \in \mathbb{R}$ is defined the same as before.

# Time-Varying Matrix Equation

Specifically, for solving time-varying matrix-inversion problem (9), we define different ZFs as below:

$$E(t) = A^{-1}(t) - X(t) \tag{12}$$

$$E(t) = A(t) - X^{-1}(t) \tag{13}$$

$$E(t) = A(t)X(t) - I, \tag{14}$$

$$E(t) = X(t)A(t) - I, \tag{15}$$

$$E(t) = (A(t)X(t))^{-1} - I, \tag{16}$$

$$E(t) = (X(t)A(t))^{-1} - I. \tag{17}$$

Before constructing different ZD models from different ZFs, we present the following theorem for further discussion.

# Time-Varying Matrix Equation

## Theorem

The time derivative of the time-varying matrix inverse $A^{-1}(t)$ is formulated as $\dot{A}^{-1}(t) = \mathrm{d}A^{-1}(t)/\mathrm{d}t = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$.

## Proof.

Since $A(t)A^{-1}(t) = I \in \mathbb{R}^{n\times n}$, we have

$$\frac{\mathrm{d}\left(A(t)A^{-1}(t)\right)}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}t} = \mathbf{0} \in \mathbb{R}^{n\times n}.$$

Expanding the above equation, we obtain

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}A^{-1}(t) + A(t)\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = \mathbf{0} \in \mathbb{R}^{n\times n},$$

which is further rewritten as

$$A(t)\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -\frac{\mathrm{d}A(t)}{\mathrm{d}t}A^{-1}(t) = -\dot{A}(t)A^{-1}(t).$$

## Time-Varying Matrix Equation

**Proof.**

Then, we have

$$\dot{A}^{-1}(t) = \frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

i.e.,

$$\dot{A}^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

□

# Time-Varying Matrix Equation

Therefore, we have following fact:

$$\frac{\mathrm{d}X^{-1}(t)}{\mathrm{d}t} = -X^{-1}(t)\dot{X}(t)X^{-1}(t) \tag{18}$$

$$\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -A^{-1}(t)\dot{A}(t)A^{-1}(t) \tag{19}$$

$$\frac{\mathrm{d}(A(t)X(t))^{-1}}{\mathrm{d}t} = -(A(t)X(t))^{-1}\frac{\mathrm{d}(A(t)X(t))}{\mathrm{d}t}(A(t)X(t))^{-1} \tag{20}$$

Considering ZD design formula (11), ZF (12), and equation (19), we have

$$\mathrm{d}A^{-1}(t) = -\gamma(A(t)X(t) - I)A(t) - \dot{A}(t), \tag{21}$$

which is also rewritten in the following explicit form:

$$\dot{X}(t) = \dot{X}(t) + (A(t)\dot{X}(t) - \gamma(A(t)X(t) - I))A(t) + \dot{A}(t)$$

Therefore, based on ZF (12), we obtain ZD model (20) for time-varying matrix inversion.

# Time-Varying Matrix Equation

Similarly, we obtain ZD models using ZFs equations (12)–(17), respectively.

| ZF | ZD model |
|---|---|
| (12) | $\dot{X}(t) = \dot{X}(t) + (A(t)\dot{X}(t) - \gamma(A(t)X(t) - I))A(t) + \dot{A}(t)$ |
| (13) | $\dot{X}(t) = -X^{-1}(t)\dot{X}(t)X^{-1}(t) - \gamma X(t)(A(t)X(t) - I)$ |
| (14) | $\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)$ |
| (15) | $\dot{X}(t) = \dot{X}(t)(I - A(t)) - X(t)\dot{A}(t) - \gamma(X(t)A(t) - I)$ |
| (16) | $\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)A(t)X(t)$ |
| (17) | $\dot{X}(t) = \dot{X}(t)(I - A(t)) - X(t)\dot{A}(t) - \gamma X(t)A(t)(X(t)A(t) - I)$ |

Table 2: Different ZFs resulting in different ZD models (depicted in explicit dynamics for modeling purposes) for time-varying matrix inversion

# Time-Varying Matrix Equation

**Theorem**

Let us consider a smoothly time-varying nonsingular matrix $A(t) \in \mathbb{R}^{n\times n}$ in (9). Starting from an initial state $X(0) \in \mathbb{R}^{n\times n}$, the state matrix $X(t)$ of ZD model (20) derived from ZF (12) globally and exponentially converges to the theoretical time-varying inverse $A^{-1}(t)$ of matrix $A(t)$.

# Time-Varying Matrix Equation

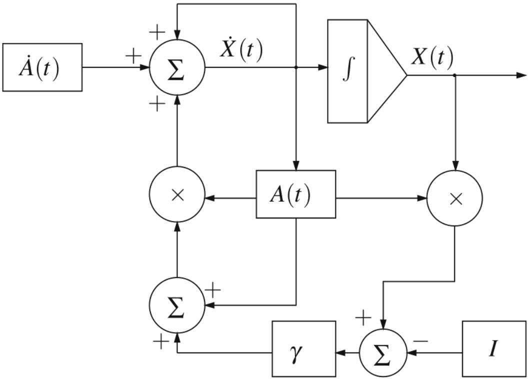


Figure 2: Block diagrams of ZD model (20) for time-varying matrix inversion

# Time-Varying Matrix Equation

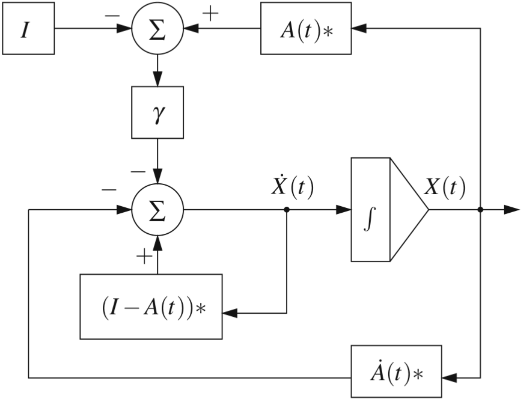

Figure 3: Block diagrams of ZD model using ZF (14) for time-varying matrix inversion

# Time-Varying Matrix Equation

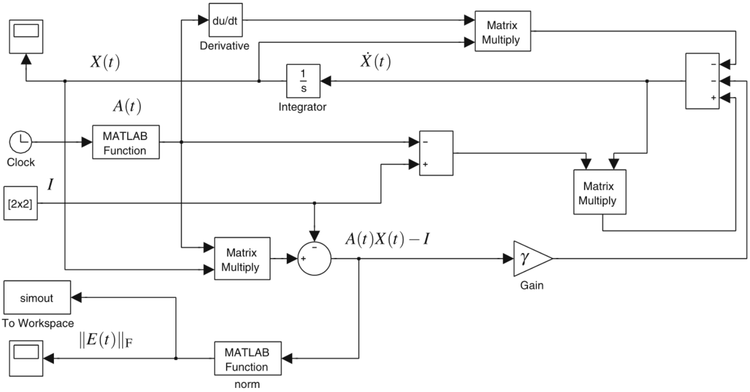


Figure 4: Overall Simulink modeling of ZD model using ZF (14) for time-varying matrix inversion

# Time-Varying Matrix Equation

## Illustrative Examples

Let us consider the time-varying matrix-inversion problem with the following time-varying matrix $A(t)$.

$$A(t) = \begin{bmatrix} \sin(5t) & \cos(5t) \\ -\cos(5t) & \sin(5t) \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad (22)$$

By algebraic operations, the theoretical time-varying inverse of $A(t)$ is given as

$$X^*(t) = A^{-1}(t) = \begin{bmatrix} \sin(5t) & -\cos(5t) \\ \cos(5t) & \sin(5t) \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad (23)$$

Thus, we can use such a theoretical solution to compare with the solutions of corresponding ZD models and then check the correctness of the models' solutions.

# Our aim(undecided)

- Generalized Sylvester Matrix Equation(GSME)

$$\begin{cases} AX - YB = C \\ DX - YE = F \end{cases} \quad (24)$$

- Generalized Linear Matrix Equation(GLME)

$$\sum_{k=1}^{n} A_k X B_k = C \quad (25)$$

- Discrete-time Algebraic Riccati Equation(DARE)

$$X = M^T XM + M^T XE(G + E^T XR)^{-1} E^T XM + C^T C \quad (26)$$

- Special case of Discrete-time Algebraic Riccati Equation

$$X = Q_1 + A_1^*(Q_2 + A_2^* X^{-1} A_2)^{-1} A_1 \quad (27)$$

or

$$X = R + M^T (X^{-1} + B)^{-1} M \quad (28)$$

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Thank you!