Mathematics, Pusan National University

LINEAR ALGEBRA AND LEARNING FROM DATA

Singular Values and Singular Vectors in the SVD

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Content

Review SVD

$A = U\Sigma V^T$ where U and V are square orthogonal matrices.

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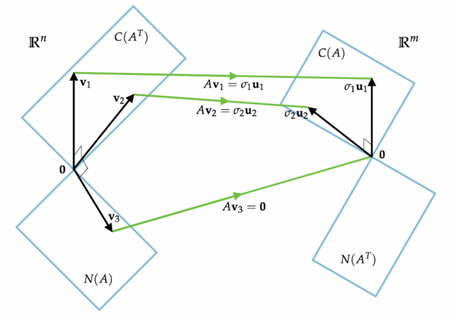

Figure: SVD and Big picture

$$A = U\Sigma V^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

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The original image needs $451 \times 439 = 197989$ pixel information, but the image can be compressed effectively through SVD. If we use only the singular values up to the 100th, we only need $100 \times (451 + 439) = 89000$ values.

Historically, the first SVD was not for vectors but for **functions**.

Example. Integral and Derivative

$$A\mathbf{x}(s) = \int_0^s \mathbf{x}(t)dt \text{ and } D\mathbf{x}(t) = \frac{d\mathbf{x}}{dt}$$

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$DA = I$ but $AD \neq I$! (For example, $f(x) = x + 1$)
D is the **pseudoinverse of** A denoted as A^+.

We can get pseudoinverse by SVD.

$$
\begin{aligned}
A^{+} &= (U\Sigma V^T)^{+} \\
&= V\Sigma^{+}U^T \\
&= \underbrace{\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r & \cdots & \mathbf{v}_n \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r \\ & & \end{bmatrix}}_{n\times m} \underbrace{\begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r & \cdots & \mathbf{u}_m \end{bmatrix}^T}_{m\times m}
\end{aligned}
$$

$$\Rightarrow A^{+}A = (V\Sigma^{+}U^T)(U\Sigma V^T) = I$$

Let A is full-column rank matrix, if there is no solution of the matrix equation $A\mathbf{x} = \mathbf{b}$, we can find the best solution $\mathbf{x}^+$ such that $A\mathbf{x}^+ = \mathbf{p}$. For example,

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \mathbf{b} \tag{1}$$

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(1) doesn't have solution, multiply both sides by A^T and project b into the column space of A.

$$A^T A\mathbf{x}^+ = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix} = A^T\mathbf{b} \Rightarrow \mathbf{x}^+ = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \tag{2}$$

$$\Rightarrow A\mathbf{x}^+ = \mathbf{p} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}, \mathbf{e} = \begin{bmatrix} -1/6 \\ 1/3 \\ -1/6 \end{bmatrix}$$

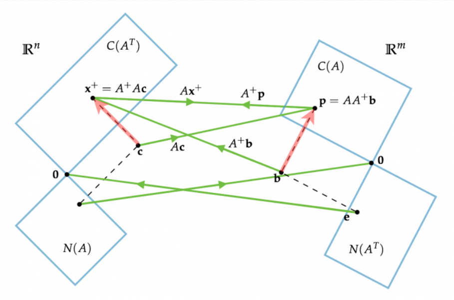

Figure: Pseudoinverse and Big picture

If we apply the pseudo inverse matrix, the solution is $\mathbf{x}^+ = A^+\mathbf{b}$ even if A is not full rank.

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Calculate pseudoinverse of A,

$$A^+ = \begin{bmatrix} -1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}^+ = A^+\mathbf{b} = \begin{bmatrix} -1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix}$$

Thank you!