Mathematics, Pusan National University

NUMERICAL LINEAR ALGEBRA

Lecture 3. Norms

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Content

Definition (Norm)

The function $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$ is called **"norm"** if $\|\cdot\|$ satisfies following three conditions. For all vectors $\mathbf{x}, \mathbf{y}$ and for all scalars $\alpha \in \mathbb{C}$,

1. $\|\mathbf{x}\| \geq 0, \|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0$,
2. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality),
3. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.

Some Norms

1. $\|\mathbf{x}\|_1 = \sum_{i=1}^{m} |x_i|$,
2. $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \sqrt{\mathbf{x}^*\mathbf{x}}$,
3. $\|\mathbf{x}\|_\infty = \sum_{i=1}^{m} |x_i|$,
4. $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p}$ for $1 \leq p < \infty$.

Weighted 2-Norm

$$\|\cdot\|_W : \mathbb{C}^n \rightarrow \mathbb{R}$$

$$\|\mathbf{x}\|_W = \left(\sum_{i=1}^{m} |w_i x_i|^2 \right)$$

where W is the diagonal matrix in which the ith entry is the weight $w_i \neq 0$.

For $m \times n$ matrix A, we define **induce matrix norm** $C = \|A\|_{(m,n)}$ is defined as the smallest number for vector norms $\|\cdot\|_m$ and $\|\cdot\|_n$

$$\|A\|_{(m,n)} = \sup_{\substack{\mathbf{x} \in \mathbb{C}^n \\ \mathbf{x} \neq 0}} \frac{\|A\mathbf{x}\|_{(m)}}{\|\mathbf{x}\|_{(n)}} = \sup_{\substack{\mathbf{x} \in \mathbb{C}^n \\ \|\mathbf{x}\|_{(n)} = 1}} \|A\mathbf{x}\|_{(m)}$$

Hölder Inequality

For any vectors $\mathbf{x}$ and $\mathbf{y}$,

$$|\mathbf{x}^*\mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$$

where $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$

Cauchy-Schwarz inequality

$p = q = 2$ in Hölder Inequality

$$|\mathbf{x}^*\mathbf{y}| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$$

Let A be $l \times m$ matrix and B an $m \times n$ matrix, for any $\mathbf{x} \in \mathbb{C}^n$,

$$\|AB\mathbf{x}\|_{(l)} \leq \|A\|_{(l,m)}\|B\mathbf{x}\|_{(m)} \leq \|A\|_{(l,m)}\|B\|_{(m,n)}\|\mathbf{x}\|_{(n)}$$

Therefore,

$$\|AB\|_{(l,n)} \leq \|A\|_{(l,m)}\|B\|_{(m,n)}$$

Matrix Norm

1. $\|A\| \geq 0$, and $\|A\| = 0$ only if $A = 0$,
2. $\|A + B\| \leq \|A\| + \|B\|$,
3. $\|\alpha A\| = |\alpha|\|A\|$.

Frobenius norm

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$
$$= \left(\sum_{j=1}^{n}\|a_j\|_2^2\right)^{1/2}$$
$$= \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$$

Theorem

For any $A \in \mathbb{C}^{m\times n}$ and unitary $Q \in \mathbb{C}^{m\times m}$,

$$\|QA\|_2 = \|A\|_2, \quad \|QA\|_F = \|A\|_F.$$

Unitarily invariant norm

A norm $\|\cdot\|$ is called **unitarily invariant norm** if $\|UAV\| = \|A\|$ for all A and for all unitary matrices U and V

Theorem [1, Cor.3.5.10]

For any unitary invariant norm,

$$\|ABC\| \leq \|A\|_2 \|B\| \|C\|_2$$

Theorem [2, Thm.7.4.9.1] [3, Thm.5]

Let $A, B \in \mathbb{C}^{m\times n}$ have SVDs with diagonal matrices $\Sigma_A, \Sigma_B \in \mathbb{R}^{m\times n}$, where the diagonal elements are arranged in nonincreasing order. Then $\|A - B\| \geq \|\Sigma_A - \Sigma_B\|$ for every unitarily invariant norm.

Condition number

The **condition number** of $A \in \mathbf{C}^{n\times n}$ is $\kappa(A) = \|A\|\|A^{-1}\|$.

For any consistent norm and $A \in \mathbf{C}^{n\times n}$

$$\rho(A) \le \|A\|.$$

For any $A \in \mathbf{C}^{n\times n}$ and $\epsilon > 0$, there is a consistent matrix norm (depending on A) such that $\|A\| \le \rho(A) + \epsilon$ [2, Lem.5.6.10.]
In particular, if $\rho(A) < 1$, there is a consistent matrix norm such that $\|A\| < 1$.

$$\rho(A) < 1 \Rightarrow \lim_{k\to\infty} A^k = 0$$

References I

[1] Horn, Roger A., Roger A. Horn, and Charles R. Johnson. Topics in matrix analysis. Cambridge university press, 1994.

[2] Horn, Roger A., and Charles R. Johnson. Matrix analysis. Cambridge university press, 2012.

[3] Mirsky, Leon. "Symmetric gauge functions and unitarily invariant norms." The quarterly journal of mathematics 11.1 (1960): 50-59.

Thank you!