On Newton Method for the Minimal Positive Solution of a System of Multi-Variable Nonlinear Matrix Equations

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Content

Introduction

In this study, we want to solve the system of nonlinear matrix equations

$$\begin{cases} A_{1,n}X_1^n + A_{1,n-1}X_2^{n-1} + \cdots + A_{1,1}X_n + A_{1,0} = 0, \\ A_{2,n}X_2^n + A_{2,n-1}X_3^{n-1} + \cdots + A_{2,1}X_1 + A_{2,0} = 0, \\ \vdots \\ A_{n,n}X_n^n + A_{n,n-1}X_1^{n-1} + \cdots + A_{n,1}X_{n-1} + A_{n,0} = 0. \end{cases} \tag{1}$$

Every matrix in (1) is in $\mathbb{R}^{p\times p}$.

Assumption

For $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$,

- $A_{i,j}$ is a positive matrix or a nonnegative irreducible matrix,
- $-A_{i,1}$ is nonsingular M-matrix,
- $A_{i,0}$ is a positive matrix.

Introduction

Set $A_j = \bigoplus_{i=1}^{n} A_{i,j} = \mathrm{diag}(A_{1,j}, A_{2,j}, \ldots, A_{n,j}), Y = \bigoplus_{i=1}^{n} X_i = \mathrm{diag}(X_1, X_2, \ldots, X_n)$

and $P = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & I_p \\ I_p & 0 & 0 & \cdots & 0 & 0 \\ \vdots & I_p & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & I_p & 0 & 0 \\ 0 & 0 & \cdots & \cdots & I_p & 0 \end{bmatrix} \in \mathbb{R}^{np \times np}$ for $j = 0, 1, \ldots, n$, then the system (1) can be rewritten as

$$\begin{aligned} \Longrightarrow F(Y) &= A_n Y^n + A_{n-1} P^\top Y^{n-1} P + \cdots + A_1 (P^\top)^{n-1} Y P^{n-1} + A_0 \\ &= \sum_{j=0}^{n} A_j (P^\top)^{n-j} Y^j P^{n-j} = 0 \end{aligned} \tag{2}$$

Introduction

Let $P(X)$ is matrix polynomial equation with degree n defined by

$$P(X) = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_0 = \sum_{j=0}^{n} A_j X^j = 0 \tag{3}$$

In [7], Seo and Kim apply Newton's method to solve (3) with the following assumptions.

Assumption

For the matrix polynomial $P(X)$ in (3),

- The coefficient matrices $A_n, A_{n-1}, \ldots, A_2$ and A_0 are nonnegative.
- $-A_1$ is nonsingular M-matrix,
- $(A_n + A_{n-1} + \cdots + A_2)\mathbf{1}_m > 0$ where $\mathbf{1}_m$ is an m-column vector with element 1.

Classical and Modified Newton's Iteration

For general, for the function $F : \mathbb{R}^{p\times p} \to \mathbb{R}^{p\times p}$, we can apply Newton's method with initial X_0 that

$$X_{i+1} = X_i - F'(X_i)^{-1}F(X_i), \quad i = 0, 1, 2, \ldots$$

where F' is the Fréchet derivative of F.

Definition 2.1 ([6])

The mapping $\mathrm{G} : \mathbb{R}^n \to \mathbb{R}^n$ is **totally or Fréchet differentiable** at $\mathbf{x}$ if the Jacobian matrix exists at $\mathbf{x}$ and

$$\lim_{h\to 0} \frac{\|\mathrm{G}(\mathbf{x}+h) - \mathrm{G}\mathbf{x} - \mathrm{G}'(\mathbf{x})h\|}{\|h\|} = 0 \tag{4}$$

We apply (4) to our matrix equation system (2). Then we can obtain the Fréchet derivative of (2) denoted as $D_Y(H)$,

$$D_Y(H) = \sum_{p=1}^{n}\left(A_p\left(P^\top\right)^{n-p}\left(\sum_{q=0}^{p-1} Y^q H Y^{p-q-1}\right)(P)^{n-p}\right) \tag{5}$$

Classical and Modified Newton's Iteration

Newton's iteration for solving equation (2) can be stated as

$$\begin{cases} D_{Y_i}(H_i) = -F(Y_i), \\ Y_{i+1} = Y_i + H_i. \end{cases} \qquad i = 1, 2, \ldots \tag{6}$$

The algorithm of Newton's iteration (6) is as follows.

Given $Y_0 = 0$, ϵ *and* $i = 0$
while $\delta < \epsilon$ **do**
  Solve H_i in equation
  $D_{Y_i}(H_i) = -F(Y_i)$
  $Y_{i+1} \leftarrow Y_i + H_i$
  $i \leftarrow i + 1$
  Calculate δ
end

Algorithm 1: Newton's iteration for equation (2)

Classical and Modified Newton's Iteration

$$
\begin{cases}
\begin{cases}
A_{1,n}\Gamma_n^{(1,i)}(H_{1,i}) + A_{1,n-1}\Gamma_{n-1}^{(2,i)}(H_{2,i}) + \cdots + A_{1,1}H_{n,i} = -F_1(X_{1,i},\ldots,X_{n,i}),\\
A_{2,n}\Gamma_n^{(2,i)}(H_{2,i}) + A_{2,n-1}\Gamma_{n-1}^{(3,i)}(H_{3,i}) + \cdots + A_{2,1}H_{1,i} = -F_2(X_{1,i},\ldots,X_{n,i}),\\
\vdots\\
A_{n,n}\Gamma_n^{(n,i)}(H_{n,i}) + A_{n,n-1}\Gamma_{n-1}^{(1,i)}(H_{1,i}) + \cdots + A_{n,1}H_{n-1,i} = -F_n(X_{1,i},\ldots,X_{n,i}),
\end{cases}\\
\begin{cases}
X_{1,i+1} = X_{1,i} + H_{1,i},\\
X_{2,i+1} = X_{2,i} + H_{2,i},\\
\vdots \qquad\qquad\qquad\qquad i = 1,2,\ldots\\
X_{n,i+1} = X_{n,i} + H_{n,i},\\
\text{where } \Gamma_k^{(j,i)}(H) = \sum_{p=1}^{k} X_{j,i}^{p-1} H X_{j,i}^{k-p} \text{ for } j = 1,2,\ldots,n.
\end{cases}
\end{cases}
\tag{7}
$$

Classical and Modified Newton's Iteration

Let $\Lambda_k^{(l)}(A_{i,j}) = \sum_{p=1}^{k} \left((X_l^{k-p})^\top \otimes A_{i,j} X_l^{p-1} \right)$, then the first n equations in (7) are equivalent to

$$
\begin{cases}
-\mathrm{vec}(F_1(X_{1,i},\ldots,X_{n,i})) \\
\quad = \Lambda_n^{(1,i)}(A_{1,n})\mathrm{vec}(H_{1,i}) + \Lambda_{n-1}^{(2,i)}(A_{1,n-1})\mathrm{vec}(H_{2,i}) + \cdots + \left(I_p \otimes A_{1,1}\right)\mathrm{vec}(H_{n,i}), \\
-\mathrm{vec}(F_2(X_{1,i},\ldots,X_{n,i})) \\
\quad = \Lambda_n^{(2,i)}(A_{2,n})\mathrm{vec}(H_{2,i}) + \Lambda_{n-1}^{(3,i)}(A_{2,n-1})\mathrm{vec}(H_{3,i}) + \cdots + \left(I_p \otimes A_{2,1}\right)\mathrm{vec}(H_{1,i}), \\
\qquad \vdots \\
-\mathrm{vec}(F_n(X_{1,i},\ldots,X_{n,i})) \\
\quad = \Lambda_n^{(n,i)}(A_{n,n})\mathrm{vec}(H_{n,i}) + \Lambda_{n-1}^{(1,i)}(A_{n,n-1})\mathrm{vec}(H_{1,i}) + \cdots + \left(I_p \otimes A_{n,1}\right)\mathrm{vec}(H_{n-1,i}).
\end{cases}
\tag{8}
$$

Classical and Modified Newton's Iteration

Let the block matrix M_i as

$$M_i = -\begin{bmatrix} I_p \otimes A_{1,1} & \Lambda_n^{(1,i)}(A_{1,n}) & \cdots & \Lambda_2^{(n-1,i)}(A_{1,2}) \\ \Lambda_2^{(n,i)}(A_{2,2}) & I_p \otimes A_{2,1} & \cdots & \Lambda_3^{(n-1,i)}(A_{2,3}) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_n^{(n,i)}(A_{n,n}) & \Lambda_{n-1}^{(1,i)}(A_{n,n-1}) & \cdots & I_p \otimes A_{n,1} \end{bmatrix}.$$

Then (8) can be rewritten as

$$M_i \begin{bmatrix} \mathrm{vec}(H_{n,i}) \\ \mathrm{vec}(H_{1,i}) \\ \vdots \\ \mathrm{vec}(H_{n-1,i}) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(F_1(X_{1,i},\ldots,X_{n,i})) \\ \mathrm{vec}(F_2(X_{1,i},\ldots,X_{n,i})) \\ \vdots \\ \mathrm{vec}(F_n(X_{1,i},\ldots,X_{n,i})) \end{bmatrix}. \tag{9}$$

Classical and Modified Newton's Iteration

The algorithm of Newton's method (7) is as follows:

Given $X_{1,0} = X_{2,0} = \cdots = X_{n,0} = 0$, ϵ *and* $i = 0$
while $\delta < \epsilon$ **do**
  Make M_i
  Solve $\mathrm{vec}(H_{j,i})$ in equation (9)
  for $j = 1$ *to* n **do**
    Reshape $H_{j,i}$ from $\mathrm{vec}(H_{j,i})$
    $X_{j,i+1} \leftarrow X_{j,i} + H_{j,i}$
  end
  $i \leftarrow i + 1$
  Calculate δ
end

Algorithm 2: Modified Newton's iteration for system (1)

Convergence of Modified Newton's Iteration

First, we use the following lemma for proof of the main theorem about convergence of modified Newton's iteration.

Lemma

Let $U, X \in \mathbb{R}^{p\times p}$, if $U > X \geq 0$, then

$$U^n - \sum_{i=1}^{n}(X^{i-1}UX^{n-i}) + (n-1)X^n > 0 \text{ for } n = 2, 3, \ldots . \qquad (10)$$

Convergence of Modified Newton's Iteration

Definition 3.1 (Z-matrix)

For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, if its off-diagonal entries are less than or equal to zero, i.e.

$$a_{ij} \leq 0 \, , \; i \neq j$$

then A is called the Z-matrix.

Definition 3.2 (M-matrix)

A matrix $A \in \mathbb{R}^{n \times n}$ is an M-matrix if $A = sI - B$ for some nonnegative matrix B and s with $s \geq \rho(B)$ where ρ is the spectral radius; it is a singular M-matrix if $s = \rho(B)$ and a nonsingular M-matrix if $s > \rho(B)$.

Convergence of Modified Newton's Iteration

Theorem 3.3

If A is the Z-matrix, then the following are equivalent:

1. A *is a nonsingular* M*-matrix.*
2. A^{-1} *is nonnegative.*
3. $Av > 0$ *for some vector* $v > 0$*.*
4. *All eigenvalues of* A *have positive real parts.*
5. $Av \geq 0$ *implies* $v \geq 0$*.*

Convergence of Modified Newton's Iteration

Theorem 3.4 (Main Theorem)

Suppose that the system of nonlinear matrix equation (1) *satisfies Assumption. Suppose that there is a collection of positive matrices* $(U_1, U_2, \ldots, U_n)$ *such that* $F_i(U_1, U_2, \ldots, U_n) \leq 0$ *for* $i = 1, 2, \ldots, n$. *Set* $X_{1,0} = X_{2,0} = \cdots = X_{n,0} = 0$, *then the sequences* $\{X_{1,i}\}, \{X_{2,i}\}, \ldots, \{X_{n,i}\}$ *generated by iteration* (7) *converge to the minimal positive solution of system* (1), *that is there is a collection of matrices* $(S_1, S_2, \ldots, S_n)$ *which is the minimal positive solution of the system* (1) *such that*

$$\lim_{i\to\infty} X_{j,i} = S_j, \text{ for } j = 1, 2, \ldots, n.$$

Convergence of Modified Newton's Iteration

Theorem 3.5 (Main Theorem (Cont.))

Moreover,

$$M_i = -\begin{bmatrix} I_p \otimes A_{1,1} & \Lambda_n^{(1,i)}(A_{1,n}) & \cdots & \Lambda_2^{(n-1,i)}(A_{1,2}) \\ \Lambda_2^{(n,i)}(A_{2,2}) & I_p \otimes A_{2,1} & \cdots & \Lambda_3^{(3,i)}(A_{2,n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_n^{(n,i)}(A_{n,n}) & \Lambda_{n-1}^{(1,i)}(A_{n,n-1}) & \cdots & I_p \otimes A_{n,1} \end{bmatrix},$$

where $\Lambda_k^{(l)}(A_{i,j}) = \sum_{p=1}^{k} \left((X_l^{k-p})^\top \otimes A_{i,j} X_l^{p-1} \right)$

is a nonsingular M-matrix for each $X_{j,i}$.

Convergence of Modified Newton's Iteration

Proof.

We use mathematical induction. Let $U_1, U_2, \ldots, U_n$ be positive matrices such that

$$\begin{cases} F_1(U_1, U_2, \ldots, U_n) \leq 0, \\ F_2(U_1, U_2, \ldots, U_n) \leq 0, \\ \qquad \vdots \\ F_n(U_1, U_2, \ldots, U_n) \leq 0. \end{cases}$$

It is equivalent to

$$\begin{cases} A_{1,n}U_1^n + A_{1,n-1}U_2^{n-1} + \cdots + A_{1,2}U_{n-1}^2 + A_{1,0} \leq -A_{1,1}U_n, \\ A_{2,n}U_2^n + A_{2,n-1}U_3^{n-1} + \cdots + A_{2,2}U_n^2 + A_{2,0} \leq -A_{2,1}U_1, \\ \qquad \vdots \\ A_{n,n}U_n^n + A_{n,n-1}U_1^{n-1} + \cdots + A_{n,2}U_{n-2}^2 + A_{n,0} \leq -A_{n,1}U_{n-1}. \end{cases} \tag{11}$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

We will show following three statements:

$$\begin{cases} X_{1,i} \leq U_1, \\ X_{2,i} \leq U_2, \\ \quad\vdots \\ X_{n,i} \leq U_n, \end{cases} \qquad M_i \text{ is a nonsingular } M\text{-matrix}, \qquad \begin{cases} X_{1,i} \leq X_{1,i+1}, \\ X_{2,i} \leq X_{2,i+1}, \\ \quad\vdots \\ X_{n,i} \leq X_{n,i+1}. \end{cases}$$

(12) (13) (14)

Since $X_{1,0} = X_{2,0} = \cdots = X_{n,0} = 0$,

$$M_0 = -\begin{bmatrix} I_p \otimes A_{1,1} & 0 & \cdots & 0 \\ 0 & I_p \otimes A_{2,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_p \otimes A_{n,1} \end{bmatrix}$$ is nonsingular M-matrix.

And since $H_{i,1} = (-A_{i,1})^{-1} A_{i,0} \geq 0$ for $i = 1, 2, \ldots, n$, the statement (12), (13), (14) are true for $i = 0$.

Convergence of Modified Newton's Iteration

Proof (Cont.)

Suppose that the statements (12)–(14) are true for $i = k \in \mathbb{N}$, we will prove three inequalities are true

$$M_k \begin{bmatrix} \mathrm{vec}(U_n - X_{n,k+1}) \\ \mathrm{vec}(U_1 - X_{1,k+1}) \\ \vdots \\ \mathrm{vec}(U_{n-1} - X_{n-1,k+1}) \end{bmatrix} \geq 0, \tag{15}$$

$$M_{k+1} \begin{bmatrix} \mathrm{vec}\left(U_n - X_{n,k+1}\right) \\ \mathrm{vec}\left(U_1 - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(U_{n-1} - X_{n-1,k+1}\right) \end{bmatrix} > 0, \tag{16}$$

$$M_{k+1} \begin{bmatrix} \mathrm{vec}\left(X_{n,k+2} - X_{n,k+1}\right) \\ \mathrm{vec}\left(X_{1,k+2} - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(X_{n-1,k+2} - X_{n-1,k+1}\right) \end{bmatrix} \geq 0. \tag{17}$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

Suppose that the statements (12)–(14) are true for $i = k \in \mathbb{N}$, we will prove three inequalities are true

$$M_k \begin{bmatrix} \mathrm{vec}(U_n - X_{n,k+1}) \\ \mathrm{vec}(U_1 - X_{1,k+1}) \\ \vdots \\ \mathrm{vec}(U_{n-1} - X_{n-1,k+1}) \end{bmatrix} \geq 0, \quad M_{k+1} \begin{bmatrix} \mathrm{vec}\left(U_n - X_{n,k+1}\right) \\ \mathrm{vec}\left(U_1 - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(U_{n-1} - X_{n-1,k+1}\right) \end{bmatrix} > 0, \quad M_{k+1} \begin{bmatrix} \mathrm{vec}\left(X_{n,k+2} - X_{n,k+1}\right) \\ \mathrm{vec}\left(X_{1,k+2} - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(X_{n-1,k+2} - X_{n-1,k+1}\right) \end{bmatrix} \geq 0.$$

(15) (16) (17)

$\Downarrow$ $\nearrow\!\!\!\!\nearrow$ $\Downarrow$ $\nearrow\!\!\!\!\nearrow$ $\Downarrow$

$$\begin{cases} X_{1,k+1} \leq U_1, \\ X_{2,k+1} \leq U_2, \\ \quad\vdots \\ X_{n,k+1} \leq U_n, \end{cases} \qquad M_{k+1} \text{ is a nonsingular } M\text{-matrix}, \qquad \begin{cases} X_{1,k+1} \leq X_{1,k+2}, \\ X_{2,k+1} \leq X_{2,k+2}, \\ \quad\vdots \\ X_{n,k+1} \leq X_{n,k+2}. \end{cases}$$

(18) (19) (20)

Convergence of Modified Newton's Iteration

Proof (Cont.)

Let $M_k^{(i)}$ is ith row partition of M_k, for example,

$$M_k^{(1)} = -\left[I_p \otimes A_{1,1} \quad \Lambda_n^{(1,k)}(A_{1,n}) \quad \cdots \quad \Lambda_2^{(n-1,k)}(A_{1,2}) \right].$$

Then it is enough to show the following inequalities instead of (18)-(20):

$$M_k^{(1)} \begin{bmatrix} \mathrm{vec}(U_n - X_{n,k+1}) \\ \mathrm{vec}(U_1 - X_{1,k+1}) \\ \vdots \\ \mathrm{vec}(U_{n-1} - X_{n-1,k+1}) \end{bmatrix} \geq 0, \; M_{k+1}^{(1)} \begin{bmatrix} \mathrm{vec}\left(U_n - X_{n,k+1}\right) \\ \mathrm{vec}\left(U_1 - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(U_{n-1} - X_{n-1,k+1}\right) \end{bmatrix} > 0, \; M_{k+1}^{(1)} \begin{bmatrix} \mathrm{vec}\left(X_{n,k+2} - X_{n,k+1}\right) \\ \mathrm{vec}\left(X_{1,k+2} - X_{1,k+1}\right) \\ \vdots \\ \mathrm{vec}\left(X_{n-1,k+2} - X_{n-1,k+1}\right) \end{bmatrix} \geq 0.$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

$$
\begin{aligned}
M_k^{(1)} \begin{bmatrix} \mathrm{vec}(U_n - X_{n,k+1}) \\ \mathrm{vec}(U_1 - X_{1,k+1}) \\ \vdots \\ \mathrm{vec}(U_{n-1} - X_{n-1,k+1}) \end{bmatrix} =& M_k^{(1)} \begin{bmatrix} \mathrm{vec}(U_n) \\ \mathrm{vec}(U_1) \\ \vdots \\ \mathrm{vec}(U_{n-1}) \end{bmatrix} - M_k^{(1)} \begin{bmatrix} \mathrm{vec}(X_{n,k+1}) \\ \mathrm{vec}(X_{1,k+1}) \\ \vdots \\ \mathrm{vec}(X_{n-1,k+1}) \end{bmatrix} \\
=& - \mathrm{vec}\left(A_{1,n}\Gamma_n^{(1,k)}(U_1) + \cdots + A_{1,2}\Gamma_2^{(n-1,k)}(U_{n-1}) + A_{1,1}U_n \right) \\
&+ \mathrm{vec}\left((n-1)A_{1,n}X_{1,k}^n + \cdots + A_{1,2}X_{n-1,k}^2 - A_{1,0} \right) \\
=& \mathrm{vec}\left(A_{1,n}\left(U_1^n - \sum_{i=1}^{n} (X_{1,k}^{i-1}U_1X_{1,k}^{n-i}) + (n-1)X_{1,k}^n \right)\right) + \cdots \\
&+ \mathrm{vec}\left(A_{1,2}\left(U_{n-1}^2 - \sum_{i=1}^{2} (X_{n-1,k}^{i-1}U_{n-1}X_{n-1,k}^{2-i}) + X_{n-1,k}^n \right)\right) \\
\geq& \ 0.
\end{aligned}
$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

$$M_{k+1}^{(1)}\begin{bmatrix}\mathrm{vec}\left(U_n-X_{n,k+1}\right)\\ \mathrm{vec}\left(U_1-X_{1,k+1}\right)\\ \vdots\\ \mathrm{vec}\left(U_{n-1}-X_{n-1,k+1}\right)\end{bmatrix}=M_{k+1}^{(1)}\begin{bmatrix}\mathrm{vec}\left(U_n\right)\\ \mathrm{vec}\left(U_1\right)\\ \vdots\\ \mathrm{vec}\left(U_{n-1}\right)\end{bmatrix}-M_{k+1}^{(1)}\begin{bmatrix}\mathrm{vec}\left(X_{n,k+1}\right)\\ \mathrm{vec}\left(X_{1,k+1}\right)\\ \vdots\\ \mathrm{vec}\left(X_{n-1,k+1}\right)\end{bmatrix}$$

$$=\mathrm{vec}\left(nA_{1,n}X_{1,k+1}^{n}+\cdots+2A_{1,2}X_{n-1,k+1}^{2}+A_{1,1}X_{n,k+1}\right)-\mathrm{vec}\left(A_{1,n}\Gamma_n^{(1,k+1)}(U_1)+\cdots+A_{1,2}\Gamma_2^{(n-1,k+1)}(U_{n-1})+A_{1,1}U_n\right)$$

$$\geq\mathrm{vec}\left(A_{1,n}\left(U_1^n-\sum_{i=1}^{n}\left(X_{1,k+1}^{i-1}U_1X_{1,k+1}^{n-i}\right)+nX_{1,k+1}^{n}\right)\right)+\mathrm{vec}\left(A_{1,n-1}\left(U_2^{n-1}-\sum_{i=1}^{n-1}\left(X_{2,k+1}^{i-1}U_2X_{2,k+1}^{n-i-1}\right)+(n-1)X_{2,k+1}^{n-1}\right)\right)$$

$$+\cdots+\mathrm{vec}\left(A_{1,2}\left(U_{n-1}^2-\sum_{i=1}^{2}\left(X_{n-1,k+1}^{i-1}U_{n-1}X_{n-1,k+1}^{2-i}\right)+2X_{n-1,k+1}^{2}\right)\right)+\mathrm{vec}\left(A_{1,1}X_{n,k+1}+A_{1,0}\right)$$

$$=\mathrm{vec}\left(A_{1,n}\left(U_1^n-\sum_{i=1}^{n}\left(X_{1,k+1}^{i-1}U_1X_{1,k+1}^{n-i}\right)+(n-1)X_{1,k+1}^{n}\right)\right)+\mathrm{vec}\left(A_{1,n-1}\left(U_2^{n-1}-\sum_{i=1}^{n-1}\left(X_{2,k+1}^{i-1}U_2X_{2,k+1}^{n-i-1}\right)+(n-1)X_{2,k+1}^{n-2}\right)\right)$$

$$+\cdots+\mathrm{vec}\left(A_{1,2}\left(U_{n-1}^2-\sum_{i=1}^{2}\left(X_{n-1,k+1}^{i-1}U_{n-1}X_{n-1,k+1}^{2-i}\right)+X_{n-1,k+1}^{2}\right)\right)+\mathrm{vec}\left(A_{1,n}X_{n,k+1}^{n}+\cdots+A_{1,1}X_{n,k+1}+A_{1,0}\right)$$

$$\geq\mathrm{vec}\left(A_{1,n}\left(U_1-X_{1,k+1}\right)\left(U_1^{n-1}-X_{1,k+1}^{n-1}\right)+A_{1,n-1}\left(U_2-X_{2,k+1}\right)\left(U_2^{n-2}-X_{2,k+1}^{n-2}\right)+\cdots+A_{1,2}\left(U_{n-1}-X_{n-1,k+1}\right)^2\right)$$

$$+\mathrm{vec}\left(A_{1,n}\left(X_{1,k+1}-X_{1,k}\right)\left(X_{1,k+1}^{n-1}-X_{1,k}^{n-1}\right)+A_{1,n-1}\left(X_{2,k+1}-X_{2,k}\right)\left(X_{2,k+1}^{n-2}-X_{2,k}^{n-2}\right)+\cdots+A_{1,2}\left(X_{n-1,k+1}-X_{n-1,k}\right)^2\right)$$

$$>0.$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

$$
\begin{aligned}
M_{k+1}^{(1)} \begin{bmatrix} \operatorname{vec}\left(X_{n,k+2} - X_{n,k+1}\right) \\ \operatorname{vec}\left(X_{1,k+2} - X_{1,k+1}\right) \\ \vdots \\ \operatorname{vec}\left(X_{n-1,k+2} - X_{n-1,k+1}\right) \end{bmatrix} &= M_{k+1}^{(1)} \begin{bmatrix} \operatorname{vec}\left(X_{n,k+2}\right) \\ \operatorname{vec}\left(X_{1,k+2}\right) \\ \vdots \\ \operatorname{vec}\left(X_{n-1,k+2}\right) \end{bmatrix} - M_{k+1}^{(1)} \begin{bmatrix} \operatorname{vec}\left(X_{n,k+1}\right) \\ \operatorname{vec}\left(X_{1,k+1}\right) \\ \vdots \\ \operatorname{vec}\left(X_{n-1,k+1}\right) \end{bmatrix} \\
&= \operatorname{vec}\left(nA_{1,n}X_{1,k+1}^{n} + \cdots + 2A_{1,2}X_{n-1,k+1}^{2} + A_{1,1}X_{n,k+1}\right) \\
&\quad - \operatorname{vec}\left((n-1)A_{1,n}X_{1,k+1}^{n} + \cdots + A_{1,2}X_{n-1,k+1}^{2} + A_{1,0}\right) \\
&= \operatorname{vec}\left(A_{1,n}X_{n,k+1}^{n} + \cdots + A_{1,1}X_{n,k+1} + A_{1,0}\right) \\
&\geq \operatorname{vec}\left(A_{1,n}\left(X_{1,k+1} - X_{1,k}\right)\left(X_{1,k+1}^{n-1} - X_{1,k}^{n-1}\right) + \cdots + A_{1,2}\left(X_{n-1,k+1} - X_{n-1,k}\right)^{2}\right) \\
&\geq 0.
\end{aligned}
$$

Convergence of Modified Newton's Iteration

Proof (Cont.)

$$\begin{cases} X_{1,i} \leq U_1, \\ X_{2,i} \leq U_2, \\ \vdots \\ X_{n,i} \leq U_n, \end{cases} \quad (12) \qquad M_i \text{ is a nonsingular } M\text{-matrix}, \quad (13) \qquad \begin{cases} X_{1,i} \leq X_{1,i+1}, \\ X_{2,i} \leq X_{2,i+1}, \\ \vdots \\ X_{n,i} \leq X_{n,i+1}. \end{cases} \quad (14)$$

Therefore, the statements (12), (13) and (14) are true for all $i \in \mathbb{N}$. It implies that the matrix sequence $X_{j,i}$ is monotonically increasing and bounded above. By Monotone convergence theorem, there are positive matrices $S_1, S_2, \ldots, S_n$ such that $\lim_{i \to \infty} X_{j,i} = S_j$ for $j = 1, 2, \ldots, n$. Moreover, for any other positive solutions $S_1', S_2', \ldots, S_n'$, since it holds that $F_i(S_1', S_2', \ldots, S_n') \leq 0$ for $i = 1, 2, \ldots, n$, we get from statement (12) that $S_j \leq S_j'$ for $j = 1, 2, \ldots, n$. Hence, $S_1, S_2, \ldots, S_n$ is the minimal positive solution of system (1).

□

Numerical Experiments

Example 1

Consider the system of equations

$$\begin{cases} A_{1,2}X_1^2 + A_{1,1}X_2 + A_{1,0} = 0, \\ A_{2,2}X_2^2 + A_{2,1}X_1 + A_{2,0} = 0. \end{cases} \tag{21}$$

Let $p = 10, 20, \dots, 100$, and $A_{j,i}$ for $i = 0, 1, 2$ and $j = 1, 2$ are $p \times p$ matrices which, in MATLAB code, are defined as

$$\begin{aligned} A_{1,2} &= \mathtt{rand(p)}, \\ A_{2,2} &= \mathtt{rand(p)}, \\ A_{1,1} &= \mathtt{rand(p)} * \mathtt{p} - \mathtt{eye(p)} * \mathtt{p\hat{~}2}, \\ A_{2,1} &= \mathtt{rand(p)} * \mathtt{p} - \mathtt{eye(p)} * \mathtt{p\hat{~}2}, \\ A_{1,0} &= \mathtt{rand(p)}, \\ A_{2,0} &= \mathtt{rand(p)}. \end{aligned}$$

Numerical Experiments

p	CPU time(sec)		Efficiency
	Algorithm 1	Algorithm 2	
10	0.0323	0.0158	51.1737%
20	0.3768	0.1073	71.5260%
30	2.5495	0.5026	80.2861%
40	11.0100	1.9496	82.2921%
50	38.3648	6.4833	83.1009%
60	103.1862	15.9109	84.5804%
70	287.6370	35.9950	87.4860%
80	926.5837	85.6463	90.7568%
90	–	164.4290	–
100	–	310.9708	–

Table 1: Comparison of CPU time

Numerical Experiments

Example 2

Consider system (1), let $n = 5, 6, 7$ and $A_{j,i}$ for $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, n$ are 5×5 matrices which are defined as

$$
\begin{aligned}
&\text{for } i = 2, 3, \ldots, n \text{ and } j = 1, 2, \ldots, n \\
&A_{j,i} = \mathtt{eye}(5) \\
&A_{j,1} = -\frac{(n-1)(n+2)}{2} * \mathtt{eye}(5) \\
&A_{j,0} = \frac{n(n-1)}{2} * \mathtt{eye}(5)
\end{aligned}
$$

Numerical Experiments

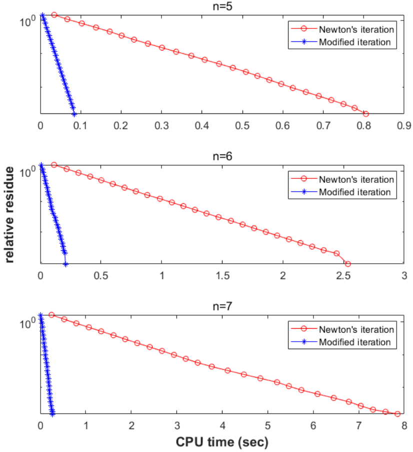

Figure 1: Comparison iteration time and relative residue

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Reference II

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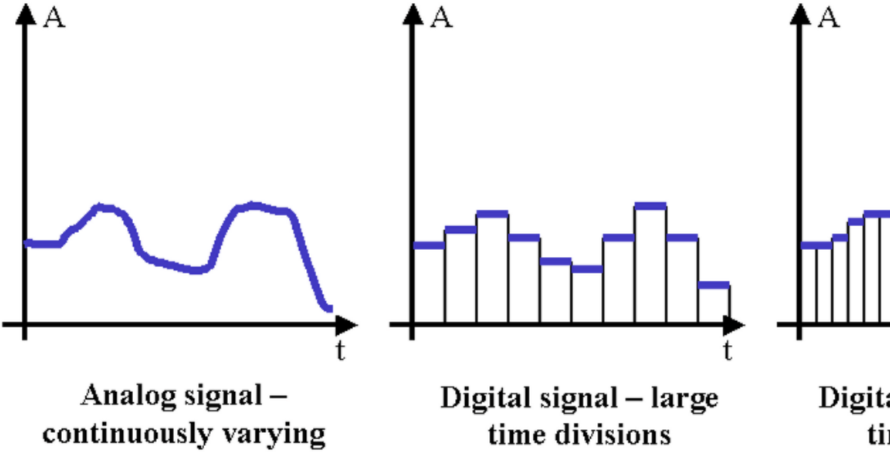
Thank you!


A
t
Analog signal – continuously varying
A
t
Digital signal – large time divisions
A