Mathematics, Pusan National University

NUMIRICAL LINEAR ALGEBRA

Projectors

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Content

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Definition

A projector is a square matrix P that satisfies

$$P^2 = P. \quad (1)$$

For $\mathbf{v} \in range(P)$, $\mathbf{v} = P\mathbf{x}$ for some $\mathbf{x}$

$$P\mathbf{v} = P^2\mathbf{v} = P\mathbf{x} = \mathbf{v}.$$

This means that $P\mathbf{v} - \mathbf{v} \in null(P)$.

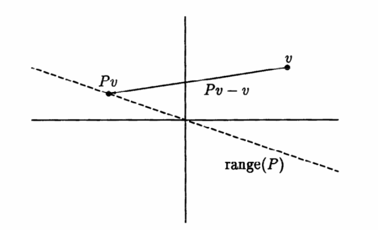


Figure: An oblique projection

If $P$is a projector, $I - P$ is also a projector,

$$(I - P)^2 = I - 2P - P^2 = I - P$$

The matrix $I - P$ is called the complementary projector to P.

$$range(I - P) = null(P) \tag{2}$$

Proof.

$\forall \mathbf{v} \in null(P), (I - P)\mathbf{v} = \mathbf{v} - P\mathbf{v} = \mathbf{v}. \Rightarrow \mathbf{v} \in range(I - P)$.
$\forall \mathbf{v}, (I - P)\mathbf{v} = \mathbf{v} - P\mathbf{v} = 0 \Rightarrow \mathbf{v} \in null(P)$. □

Also, it is easy to see that $null(I - P) = range(P)$.
And $range(P) \cap null(P) = 0$

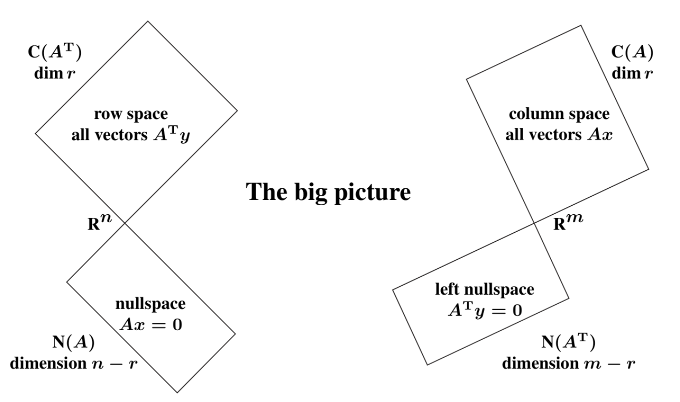


Figure: The Big Picture of Linear Algebra[2]

Let $S_1 = range(P), S_2 = null(P)$, P is orthogonal projector if $S_1 \perp S_2$.

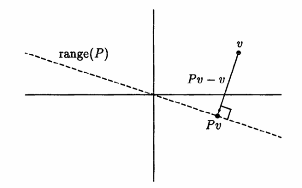


Figure: An orthogonal projection

Theorem

A projector P is orthogonal if and only if $P = P^*$

Proof.

If $P = P^*$, then the inner product between a vector $P\mathbf{x} \in S_1$ and a vector $(I - P)\mathbf{y} \in S_2$ is zero:

$$\mathbf{x}^* P^* (I - P)\mathbf{y} = \mathbf{x}^* (P - P^2)\mathbf{y} = 0$$

Thus the projector is orthogonal.

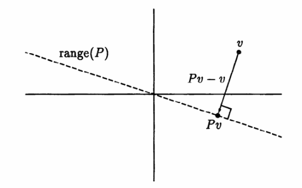


Figure: An orthogonal projection

Proof (Cont.)

We use SVD to show the "only if". Suppose that P is orthogonal. And S_1 has dimension n. Then an SVD of P can be expressed as follows. Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_m\}$ be an orthogonal basis of $\mathbb{C}^m$, where $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ is a basis of S_1 and $\{\mathbf{q}_{n+1}, \ldots, \mathbf{q}_m\}$ is a basis of S_2. For $j \leq n$, $P\mathbf{q}_j = \mathbf{q}_j$, and for $j > n$, $P\mathbf{q}_j = 0$. Let Q be the unitary matrix such that jth column is $\mathbf{q}_j$.

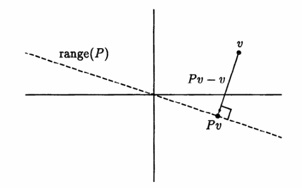


Figure: An orthogonal projection

Proof (Cont.)

$$PQ = \begin{bmatrix} q_1 & \cdots & q_n & 0 & \cdots \end{bmatrix}$$

so that

$$Q^*PQ = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} = \Sigma$$

a diagonal matrix with ones in the first n entries and zeros everywhere else.

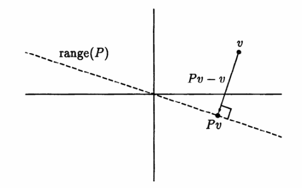


Figure: An orthogonal projection

Proof.

Thus we have constructed a singular value decomposition of P:

$$P = Q\Sigma Q^*$$

Since $P^* = (Q\Sigma Q^*)^* = Q^*\Sigma^* Q = Q\Sigma Q^* = P$, $P = P^*$ □

Since an orthogonal projector has some singular values equal to zero, We will use reduced SVD. We obtain the marvelously simple expression

$$P = \hat{Q}\hat{Q}^* \tag{3}$$

where the columns of $\hat{Q}$ are orthonormal. Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ be an any set n orthogonal vectors in $\mathbb{C}^m$, and $\hat{Q}$ be the corresponding $m \times n$ matrix. Consider $\mathbf{v}$ can be expressed as

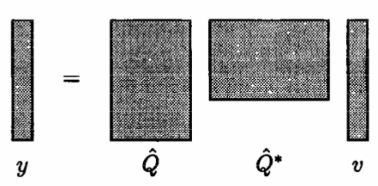
$$\mathbf{v} = \mathbf{r} + \sum_{i=1}^{n} (\mathbf{q}_i \mathbf{q}_i^*)\mathbf{v}$$

So the map

$$\mathbf{v} \mapsto \sum_{i=1}^{n} (\mathbf{q}_i \mathbf{q}_i^*)\mathbf{v} \tag{4}$$

is an orthogonal projector onto $range(P)$.

It can be written $\mathbf{y} = \hat{Q}\hat{Q}^*\mathbf{v}$:



The complement of an orthogonal projector is also an orthogonal projector.
($\because$ $I - \hat{Q}\hat{Q}^*$ is hermitian.)
The complement projects onto the space orthogonal to $range(\hat{Q})$.

Example

An important special case of orthogonal projectors is the rank-one orthogonal projector that isolates the component in a single direction $\mathbf{q}$, which can be written

$$P^* = \mathbf{q}\mathbf{q}^*. \tag{5}$$

These are the pieces from which higher-rank projectors can be made, as in (4). Their complements are the rank m - 1 orthogonal projectors that eliminate the component in the direction of $\mathbf{q}$:

$$P_{\perp \mathbf{q}} = I - \mathbf{q}\mathbf{q}^*.$$

For arbitrary nonzero vectors $\mathbf{a}$, the analogous formulas are

$$P_{\mathbf{a}} = \frac{\mathbf{a}\mathbf{a}^*}{\mathbf{a}^*\mathbf{a}}, \; P_{\perp a} = I - \frac{\mathbf{a}\mathbf{a}^*}{\mathbf{a}^*\mathbf{a}}. \tag{6}$$

Suppose that the subspace is spanned by the linearly independent vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, and let A be the $m \times n$ matrix whose jth column is $\mathbf{a}_j$. In passing from $\mathbf{v}$ to its orthogonal projection $\mathbf{y} \in range(A)$, the difference $\mathbf{y} - \mathbf{v}$ must be orthogonal to $range(A)$.

$$\mathbf{a}_j^*(\mathbf{y} - \mathbf{v}) = 0 \text{ for every } j$$

$$\begin{aligned}
\mathbf{y} \in range(A) &\Rightarrow \exists \mathbf{x} \text{ such that } \mathbf{y} = A\mathbf{x} \\
&\Rightarrow \mathbf{a}_j^*(A\mathbf{x} - \mathbf{v}) = 0 \text{ for every } j \\
&\Rightarrow A^*(A\mathbf{x} - \mathbf{v}) = 0 \\
&\Rightarrow A^*A\mathbf{x} = A^*\mathbf{v} \\
&\Rightarrow \mathbf{x} = (A^*A)^{-1}A^*\mathbf{v} \qquad (7) \\
&\Rightarrow P = A(A^*A)^{-1}A^*(\because \mathbf{y} = A\mathbf{x}) \qquad (8)
\end{aligned}$$

Projector in other books

- Matrix Computations [3]
- Matrix analysis and applied linear algebra [4]
- Linear algebra and its application [5]
- Functions of Matrices [6]
- Matrix Analysis [7]

There are several important orthogonal projections associated with the singular value decomposition. Suppose $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ is SVD of A and that $r = rank(A)$. If we have the U and V partitionings

$$U = \underset{\quad r \qquad m-r}{\begin{bmatrix} U_r & \tilde{U}_r \end{bmatrix}}, \qquad V = \underset{\quad r \qquad n-r}{\begin{bmatrix} V_r & \tilde{V}_r \end{bmatrix}}$$

Then

$$\begin{aligned}
V_r V_r^T &= \text{projection on to } null(A)^\perp = range(A^T) \\
\tilde{V}_r \tilde{V}_r^T &= \text{projection on to } null(A) \\
U_r U_r^T &= \text{projection on to } range(A) \\
\tilde{U}_r \tilde{U}_r^T &= \text{projection on to } range(A)^\perp = null(A^T)
\end{aligned}$$

Projectors

Let $\mathcal{X}$ and $\mathcal{Y}$ be complementary subspaces of a vector space $\mathcal{V}$ so that each $\mathbf{v} \in \mathcal{V}$ can be uniquely resolved as $\mathbf{v} = \mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$. The unique linear operator $\mathbf{P}$ defined by $\mathbf{Pv} = \mathbf{x}$ is called the ***projector onto*** $\mathcal{X}$ ***along*** $\mathcal{Y}$, and $\mathbf{P}$ has the following properties.

- $\mathbf{P}^2 = \mathbf{P}$ ($\mathbf{P}$ is idempotent). (5.9.8)
- $\mathbf{I} - \mathbf{P}$ is the complementary projector onto $\mathcal{Y}$ along $\mathcal{X}$. (5.9.9)
- $R(\mathbf{P}) = \{\mathbf{x} \mid \mathbf{Px} = \mathbf{x}\}$ (the set of "fixed points" for $\mathbf{P}$). (5.9.10)
- $R(\mathbf{P}) = N(\mathbf{I} - \mathbf{P}) = \mathcal{X}$ and $R(\mathbf{I} - \mathbf{P}) = N(\mathbf{P}) = \mathcal{Y}$. (5.9.11)
- If $\mathcal{V} = \Re^n$ or $\mathcal{C}^n$, then $\mathbf{P}$ is given by

$$\mathbf{P} = [\mathbf{X} \mid \mathbf{0}][\mathbf{X} \mid \mathbf{Y}]^{-1} = [\mathbf{X} \mid \mathbf{Y}] \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} [\mathbf{X} \mid \mathbf{Y}]^{-1}, \tag{5.9.12}$$

where the columns of $\mathbf{X}$ and $\mathbf{Y}$ are respective bases for $\mathcal{X}$ and $\mathcal{Y}$. Other formulas for $\mathbf{P}$ are given on p. 634.

Figure: Projectors in Matrix analysis and applied linear algebra

Angle between Complementary Subspaces.

Let angle of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, denoted as θ, is defined to be the number $0 \leq \theta \leq \pi/2$ such that

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{u}}{\|\mathbf{v}\|_2\|\mathbf{u}\|_2}.$$

How about "angle between subspaces of $\mathbb{R}^n$"?
Consider $\mathcal{R}$ and $\mathcal{N}$ such that the nonzero subspaces and $\mathbb{R}^n = \mathcal{R} \oplus \mathcal{N}$.
The angle between $\mathcal{R}$ and $\mathcal{N}$ is defined to be the number $0 \leq \theta \leq \pi/2$ such that

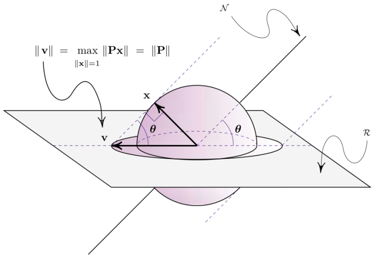
$$\cos\theta = \max_{\substack{\mathbf{u}\in\mathcal{R} \\ \mathbf{v}\in\mathcal{N}}} \frac{\mathbf{v}^T\mathbf{u}}{\|\mathbf{v}\|_2\|\mathbf{u}\|_2} = \max_{\substack{\mathbf{u}\in\mathcal{R},\mathbf{v}\in\mathcal{N} \\ \|\mathbf{u}\|_2=\|\mathbf{v}\|_2=1}} \mathbf{v}^T\mathbf{u}$$

This is a good definition, but it's not easy to compute.

Angle between Complementary Subspaces.

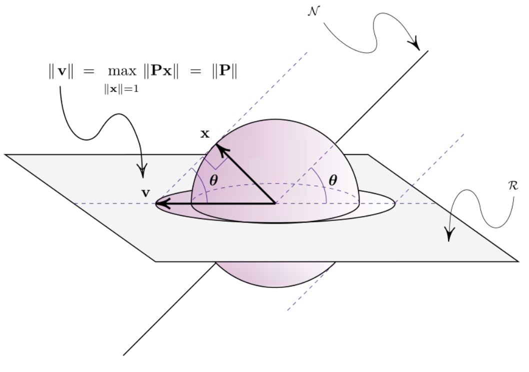
Let P be the projector such that $range(P) = \mathcal{R}$ and $null(P) = \mathcal{N}$.
Recall the matrix 2-norm of P is

$$\|P\|_2 = \max_{\|\mathbf{x}\|_2=1} \|P\mathbf{x}\|_2$$



In other words, $\|P\|_2$ is the length of a longest vector in the image of the unit sphere under transformation by P. Consider the situation in $\mathbb{R}^3$. The image of the unit sphere under P is obtained by projecting the sphere onto $\mathcal{R}$ along lines parallel to $\mathcal{N}$.

Angle between Complementary Subspaces.



The norm of a longest vector $\mathbf{v}$ on this ellipse equals the norm of P. That is, $\|v\|_2 = \max_{\|x\|_2=1} \|Px\|_2 = \|P\|_2$.
Therefore,

$$\sin\theta = \frac{\|\mathbf{x}\|_2}{\|\mathbf{v}\|_2} = \frac{1}{\|\mathbf{v}\|_2} = \frac{1}{\|P\|_2}$$

There is example of projection(projector) matrix.

Least-Squares Fitting of Data

The cost of producing t books like this one is nearly linear, $\mathbf{b} = c + d\mathrm{t}$, with editing and typesetting in c and then printing and binding in d. c is the set-up cost and d is the cost for each additional book.

$$\begin{aligned} c + dt_1 &= b_1 \\ c + dt_2 &= b_2 \\ &\vdots \\ c + dt_m &= b_m \end{aligned} \quad \Rightarrow \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad or \quad A\mathbf{x} = \mathbf{b}$$

The best solution $(\hat{c}, \hat{d})$ is the $\hat{\mathbf{x}}$ that minimizes the squared error E^2:

$$\textbf{Minimize} \quad E^2 = \|\mathbf{b} - A\mathbf{x}\| = (b_1 - C - Dt_1)^2 + \cdots + (b_m - C - Dt_m)^2.$$

Example

Let $(t_1, b_1) = (-1, 1), (t_2, b_2) = (1, 1)$ and $(t_3, b_3) = (2, 3)$, then

$$A\mathbf{x} = \mathbf{b} \quad \text{is} \quad \begin{aligned} c - d &= 1 \\ c + d &= 1 \\ c + 2d &= 3 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Since there is no straight line through all three points, they are solved by least squares:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

The best solution is $\hat{c} = \frac{9}{7}, \hat{d} = \frac{4}{7}$ and the best line is $\frac{9}{7} + \frac{4}{7}t$.
The line $\frac{9}{7} + \frac{4}{7}t$ has $\frac{5}{7}, \frac{13}{7}, \frac{17}{7}$ at $-1, 1, 2$. This vector is the projection.$(P = A(A^T A)^{-1} A^T)$

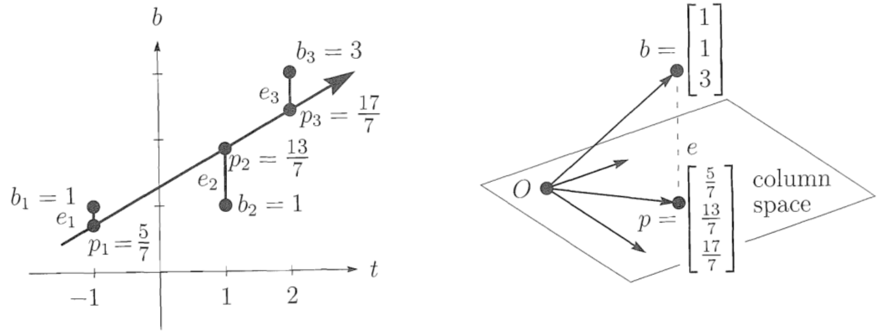


Figure: Example of straight-line approximation matches the projection p of b

The pseudoinverse $X \in \mathbb{C}^{m\times m}$ of $A \in \mathbb{C}^{m\times n}$ is the unique matrix satisfying the four Moore–Penrose conditions:

$$\begin{array}{llll} \text{(i)} & AXA = A, & \text{(ii)} & XAX = X, \\ \text{(iii)} & AX = (AX)^*, & \text{(iv)} & XA = (XA)^*. \end{array} \tag{9}$$

The pseudoinverse is denoted by A^+.
Let S be a subspace of $\mathbb{C}^m$, and let $P_S \in \mathbb{C}^{m\times m}$.

- P_S is is the projector onto S if $range(P_S) = S$ and $P_S^2 = PS$.
  The projector is not unique.
- P_S is is the orthogonal projector onto S if $range(P_S) = S$ and $P_S^2 = P_S$ and $P_S^* = P_S$.
  The orthogonal projector is unique.
  In terms of the pseudoinverse, $P_{range(A)} = AA^+$ and $P_{range(A^*)} = A^+A$.

Theorem

Let $A \in M_n$ be given. There exists a unique monic polynomial $q_A(t)$ of minimum degree that annihilates A. The degree of $q_A(t)$ is at most n. If $p(t)$ is any monic polynomial such that $p(A) = 0$, then $q_A(t)$ divides $p(t)$, that is, $p(t) = h(t)q_A(t)$ for some monic polynomial $h(t)$.

Definition

Let $A \in M_n$ be given. The unique monic polynomial $q_A(t)$ of minimum degree that annihilates A is called the minimal polynomial of A.

Example

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the minimal polynomial of A is $q_A(t) = t^2 - 1$.

Definition

Thought of as a formal polynomial in t, the characteristic polynomial of $A \in Mn$ is

$$p_A(t) = \det(tI - A)$$

We refer to the equation $p_A(t) = 0$ as the characteristic equation of A.

Example

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the characteristic polynomial of A is $p_A(t) = t^3 - t^2 - t + 1$.
And since $p_A(A) = 0$, $p_A(t) = (t-1)(t^2-1) = h(t)q_A(t)$ where $h(t) = t - 1$ and $q_A(t) = t^2 - 1$.

Theorem

Let $A \in M_n$ be given. There is a nonsingular $S \in M_n$, positive integers q and $n_1, \ldots, n_q$ with $n_1 + \cdots + n_q = n$ and scalars $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$ such that

$$A = S \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_q}(\lambda_q) \end{bmatrix} S^{-1} \tag{10}$$

The Jordan matrix $J_A = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_q}(\lambda_q)$ is uniquely determined by A up to permutation of its direct summands. If A is real and has only real eigenvalues, then S can be chosen to be real.

Fact : J_A is similar to A.(has same eigenvalues)

We may assume that $J = J_{m_1}(\lambda) \oplus \cdots \oplus J_{m_p}(\lambda) \oplus \hat{J}$, in which the Jordan matrix $\hat{J}$ is a direct sum of Jordan blocks with eigenvalues different from λ. Then

$$\begin{aligned} J - \lambda I =& (J_{m_1}(\lambda) - \lambda I) \cdots (J_{m_p}(\lambda) - \lambda I) \oplus (\hat{J} - \lambda I) \\ =& J_{m_1}(0) \oplus \cdots \oplus J_{m_p}(0) \oplus (\hat{J} - \lambda I) \end{aligned}$$

is similar to $A - \lambda I$. Moreover, $(J - \lambda I)^k$ is similar to $(A - \lambda I)^k$. And rank is similarity invariant,

$$rank(A - \lambda I)^k = rank(J - \lambda I)^k = rank J_{m_1}(0)^k + \cdots + rank J_{m_p}(0)^k + rank(\hat{J} - \lambda I)^k \qquad (11)$$

Now let $A \in M_n$, let $\lambda \in \mathbb{C}$, let k be a positive integer, let

$$r_k(A, \lambda) = rank(A - \lambda I)^k, \quad r_0(A, \lambda) := n \qquad (12)$$

and define

$$w_k(A, \lambda) = r_{k-1}(A, \lambda) - r_k(A, \lambda), \quad w_1(A, \lambda) := n - r_1(A, \lambda) \qquad (13)$$

Using (11) and (12), we can explain the algebraic meaning of $w_k(A,\lambda)$:

$$\begin{aligned} w_k(A,\lambda) =& (rank J_{m_1}(0)^{k-1} - rank J_{m_1}(0)^k) + \cdots + (rank J_{m_p}(0)^{k-1} - rank J_{m_p}(0)^k) \\ =& (1 \text{ if } m_1 \geq k) + \cdots + (1 \text{ if } m_p \geq k) \\ =& \text{number of blocks with eigenvalue } \lambda \text{ that have size at least } k \end{aligned} \tag{14}$$

Definition

The Weyr characteristic of $A \in M_n$ associated with $\lambda \in \mathbb{C}$ is

$$w(A,\lambda) = (w_1(A,\lambda), \ldots, w_q(A,\lambda))$$

in which the sequence of integers $w_j(A,\lambda)$ is defined by (13)

Definition

Let $\lambda \in \mathbb{C}$ be given, let $q \geq 1$ be a given positive integer, let $w_1 \geq \cdots \geq w_q \geq 1$ be a given nonincreasing sequence of positive integers, and let $w = (w_1, \ldots, w_q)$. The Weyr block $W(w, \lambda)$ associated with λ and w is the upper triangular $q \times q$ block bidiagonal matrix

$$W(w,\lambda) = \begin{bmatrix} \lambda I_{w_1} & G_{w_1,w_2} & & & \\ & \lambda I_{w_2} & G_{w_2,w_3} & & \\ & & \ddots & \ddots & \\ & & & \ddots & G_{w_{q-1},w_q} \\ & & & & I_{w_q} \end{bmatrix} \quad (15)$$

in which $G_{w_i,w_j} = \begin{bmatrix} I_{w_j} \\ 0 \end{bmatrix} \in M_{w_i,w_j}, \quad 1 \leq i < j$

For any $A \in M_n$, let q be the index of an eigenvalue λ of A, let $w_k = w_k(A, \lambda)$, $k = 1, 2, \ldots$ be the Weyr characteristic of A associated with λ. Define the Weyr block of A associated with the eigenvalue λ to be

$$W_A(\lambda) = W(w(A, \lambda), \lambda)$$

Theorem

Let $\lambda_1, \ldots, \lambda_d$ is be distinct eigenvalues of $A \in M_n$.

- *There is a nonsingular $S \in M_n$ and there are Weyr blocks $W_1, \ldots, W_d$ such that*
- $A = S(W_1 \oplus \cdots \oplus W_d)S^{-1}$
- *The Weyr matrix $W_1 \oplus \cdots \oplus W_d$ is similar to A.*

where $W_j = W_A(\lambda_j)$ for each $j = 1, \ldots, d$.

Let $W_A = W_A(\lambda_1) \oplus \cdots \oplus W_A(\lambda_d)$, then $A = SW_AS^{-1}$ is Weyr canonical form of A.

Example

Consider the Jordan matrix $J = J_3(\lambda) \oplus J_2(\lambda)$, then $w_J(J, \lambda) = 2, 2, 1$:

$$J = \left[\begin{array}{ccc|cc} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & & \\ 0 & 0 & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & 0 & \lambda \end{array}\right] \Rightarrow W_J(\lambda) = \left[\begin{array}{cc|cc|c} \lambda & 0 & 1 & 0 & \\ 0 & \lambda & 0 & 1 & \\ \hline & & \lambda & 0 & 1 \\ & & 0 & \lambda & 0 \\ \hline & & & & \lambda \end{array}\right]$$

The eigenvalue of J is $(\lambda, \lambda, \lambda, \lambda, \lambda)$, considering multiplicity and there are two eigenvectors corresponding to the eigenvalue. The cycle for the first eigenvector of $(J - \lambda I)$ is 3 and the second eigenvector is 2.

Theorem

Let $\lambda_1, \ldots, \lambda_d$ be the distinct eigenvalues of a given $A \in M_n$ in any prescribed order, let $q_1 \ldots, q_d$ be their respective indices, and let $q = q_1 + \cdots + q_d$. Then A is unitarily similar to an upper triangular matrix of the form

$$F = \begin{bmatrix} \mu_1 I_{n_1} & F_{12} & F_{13} & \cdots & F_{1p} \\ & \mu_2 I_{n_2} & F_{23} & \cdots & F_{2p} \\ & & \mu_1 I_{n_1} & \ddots & \vdots \\ & & & \ddots & F_{p-1\,p} \\ & & & & \mu_p I_{n_p} \end{bmatrix}$$

in which

(a) $\mu_1 = \cdots = \mu_{q_1} = \lambda_1; \mu_{q_1+1} = \cdots = \mu_{q_1+q_2} = \lambda_2; \ldots; \mu_{p-q_d+1} = \cdots = \mu_p = \lambda_d$

(b) *For each* $j = 1, \ldots, d$ *the* q_j *integers* $n_i, \ldots, n_{i+q_j-1}$ *for which* $\mu_i = \cdots = \mu_{i+q_j-1} = \lambda_j$ *are the Weyr characteristic of* λ_j *as an eigenvalue of* A, *that is,* $n_i = w_1(A, \lambda_j) \geq \cdots \geq n_{i+q_j-1} = w_{q_j}(A, \lambda_j)$

(c) *if* $\mu_i = \mu_{i+1}$ *then* $n_i \geq n_{i+1}$, $F_{i,i+1} \in M_{n_i, n_{i+1}}$ *is upper triangular, and its diagonal entries are real and positive*

Corollary

Let $A \in M_n$ be a projector: $A^2 = A$. Let

$$\sigma_1 \geq \cdots \geq \sigma_g > 1 \geq \sigma_{g+1} \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots$$

be the singular values of A, so $r = \operatorname{rank} A$ and g is the number of singular values of A that are greater than 1. Then A is unitarily similar to

$$\begin{bmatrix} 1 & (\sigma_1^2 - 1)^{1/2} \\ 0 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & (\sigma_g^2 - 1)^{1/2} \\ 0 & 0 \end{bmatrix} \oplus I_{r-g} \oplus 0_{n-r-g}$$

Reference

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Thank you!