Mathematics, Pusan National University

Numerical Linear Algebra

Lecture 14. Stability

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Content

Algorithms

In Lecture 12, we defined a mathematical *problem* as a function $f : X \to Y$ from a vector space X of data to a vector space Y of solutions.
An *algorithm* can be viewed as another map $\tilde{f} : X \to Y$ between the same two spaces.

Algorithm

Let f be a problem, a computer whose floating point system satisfies **Fundamental Axiom of Floating Poinf Arithmetic**. Given data $x \in X$, let x be rounded to floating point. Then the result of program is a collection of floating point numbers belongs to the vector space Y. Let this computed result be called $\tilde{f}(x)$.

Fundamental Axiom of Floating Poinf Arithmetic

For all $x, y \in F$, there exists ϵ with $|\epsilon| \leq \epsilon_{\text{machine}}$ such that

$$x \circledast y = (x * y)(1 + \epsilon).$$

Accuracy

Error

absolute error : $\|\tilde{f}(x) - f(x)\|$

relative error : $\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$

If $\tilde{f}$ is a good algorithm, $\begin{cases} \text{relative error is small} \\ \tilde{f} \text{ is accurate if for each } x \in X, \frac{\|\tilde{f}(x)-f(x)\|}{\|f(x)\|} = O(\epsilon_{\text{machine}}) \end{cases}$

Stabilty

Instead of aiming for accuracy in all cases, the most it is appropriate to aim for in general is stability.

Stabilty

We say that an algorithm $\tilde{f}$ for a problem f is *stable* if for each $x \in X$,

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\epsilon_{\text{machine}})$$

for some $\tilde{x}$ with

$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon_{\text{machine}}).$$

In words,

A stable algorithm gives nearly the right answer to nearly the right question.

Stabilty

The motivation for this definition will become clear in the next lecture and in applications throughout the remainder of this book.

Caution

The definitions of stability given here are useful in many parts of numerical linear algebra, the condition $O(\epsilon_{\text{machine}})$ is probably too strict to be appropriate for all numerical problems in other areas such as differential equations.

Backward Stability

Many algorithms of numerical linear algebra satisfy a condition that is both stronger and simpler than stability.

Backward Stability

An algorithm $\tilde{f}$ for a problem f is *backward stable* if for each $x \in X$,

$$\tilde{f}(x) = f(\tilde{x}) \text{ for some } \tilde{x} \text{ with } \frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon_{\text{machine}})$$

In words,

A backward stable algorithm gives exactly the right answer to nearly the right question.

The Meaning of $O(\epsilon_{\text{machine}})$

The notation

$$\phi(t) = O(\psi(t))$$

is a standard one in mathematics, with a precise definition. This equation asserts that there exist some positive constant C such that, for all t sufficiently close to an understood limit (e.g. $t \to 0$ or $t \to \infty$),

$$|\phi(t)| \leq C\psi(t)$$

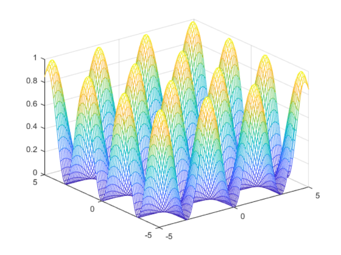
The Meaning of $O(\epsilon_{\text{machine}})$

Also standard in mathematics are statements of the from

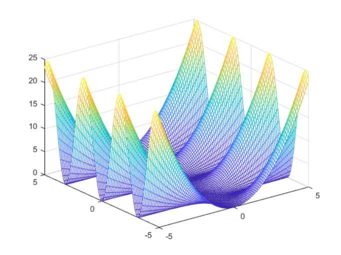
$$\phi(s,t) = O(\psi(t)) \quad \text{uniformly in } s,$$

where ϕ is a function that depends on t and s. And the word "uniformly" indicates that there exists a single constant C that holds for all choices of s.

Example



The graph of $(\sin^2 t)(\sin^2 s)$



The graph of $(\sin^2 t)s^2$

The Meaning of $O(\epsilon_{\text{machine}})$

In this book, our use of the "O" symbol follows there standard definitions. Specifically, we often state results along the lines of

$$\|\text{computed quantity}\| = O(\epsilon_{\text{machine}})$$

Here is means.

1. "$\|$computed quantity$\|$" represents the norm of some number or collection of numbers determined by algorithm $\tilde{f}$ for a problem f, depending both on the data $x \in X$ for f and on $\epsilon_{\text{machine}}$.
2. The implicit limit process is $\epsilon_{\text{machine}} \to 0$.
3. The "O" applies uniformly for all data $x \in X$.

Dependence on m and n, not A and b

Suppose we are considering an algorithm for solving a nonsingular $m \times m$ system of equations $Ax = b$ for x, and we assert that the computed result $\tilde{x}$ for this algorithm satisfies

$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\kappa(A)\epsilon_{\text{machine}}) \quad \Rightarrow \quad \frac{\|\tilde{x} - x\|}{\|x\|} \leq C\kappa(A)\epsilon_{\text{machine}} \text{ for constant } C.$$

And this means that

$$\|\tilde{x} - x\| \leq C\kappa(A)\epsilon_{\text{machine}}\|x\|.$$

In general, the constant C depends on dimension m.

Independence of Norm

Theorem

For problems f and algorithm $\tilde{f}$ defined on finite-dimensional spaces X and Y, the properties of accuracy, stability, and backward stability all hold or fail to hold independently of the choice of norms in X and Y.

Proof.

In finite-dimensional vector space, all norms are equivalent in the sense that if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on the same space, then there exist positive constant C_1 and C_2 such that $C_1\|x\| \le \|x\|' \le C_2\|x\|$. It follows that a change of norm may affect the size of the constant C implicit in a statement involving $O(\epsilon_{\text{machine}})$, but not the existence of such a constant. □

Thank you!