Mathematics, Pusan National University

Numerical Linear Algebra

Lecture 12. Conditioning and Condition Numbers

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Content

Condition of a Problem

We can view a **problem** as a function $f : X \longrightarrow Y$ from a normed vector space X of data to a normed vector space Y of solutions. **well-conditioned** problem : **small** change of x leads to **small** change in $f(x)$. **ill-conditioned** problem : **small** change of x leads to **large** change in $f(x)$

Absolute Condition Number

Let δx : small change of x, write $\delta f = f(x+\delta x) - f(x)$. **Absolute Condition Number** $\hat{\kappa} = \hat{\kappa}(x)$ of the problem f at x is defined as

$$\hat{\kappa} = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{\|\delta f\|}{\|\delta x\|}, \tag{1}$$

simply,

$$\hat{\kappa} = \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|}. \tag{2}$$

If f is diffierentiable, let $J(x)$ be a **Jacobian** of f at x. Then the absolute condition number becomes

$$\hat{\kappa} = \|J(x)\|. \tag{3}$$

Relative Condition Number

The relative condition number $\kappa = \kappa(x)$ is defined by

$$\kappa = \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right), \tag{4}$$

or, assuminf δx and δf are infinitesimal,

$$\kappa = \sup_{\|\delta x\| \leq \delta} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right). \tag{5}$$

If f is differentiable, we can express this quantity in terms of the Jacobian:

$$\kappa = \frac{\|J(x)\|}{\|f(x)\|/\|x\|}. \tag{6}$$

Examples

Example

Let $f(x) = \frac{x}{2}$ where $x \in \mathbf{C}$. The Jacobian of the f is $J = f' = \frac{1}{2}$, so,

$$\kappa = \frac{\|J\|}{\|f(x)\|/\|x\|} = \frac{1/2}{(x/2)/x} = 1. \quad (7)$$

This problem is **well-conditioned**.

Example

Consider the problem $f(x) = x_1 - x_2$ where the vector $x \in \mathbb{C}^2$. For simplicity, we use the ∞-norm on the data space $\mathbb{C}^2$. The Jacobian of f is

$$J = \begin{bmatrix} \frac{\delta f}{\delta x_1} & \frac{\delta f}{\delta x_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{8}$$

with $\|J\|_\infty = 2$. The condition number is

$$\kappa = \frac{\|J\|_\infty}{\|f(x)\|/\|x\|} = \frac{2}{|x_1 - x_2|/\max\{|x_1|, |x_2|\}}. \tag{9}$$

As $|x_1 - x_2| = 0$, the problem is **ill-conditioned**.

Condition of Matrix-Vector Multiplication

Fixed $A \in \mathbf{C}^{m\times n}$ and consider the problem of computing Ax from input x. Then,

$$\kappa = \sup_{\delta x}\left(\frac{\|A(x+\delta x)-Ax\|}{\|Ax\|}\bigg/\frac{\|\delta x\|}{\|x\|}\right) = \sup_{\delta x}\frac{\|A\delta x\|}{\|\delta x\|}\bigg/\frac{\|Ax\|}{\|x\|} = \|A\|\frac{\|x\|}{\|Ax\|} \tag{10}$$

We can use the fact that $\frac{\|x\|}{\|Ax\|} \leq \|A^{-1}\|$ to lossen (10) to a bound indepenfent of x:

$$\kappa \leq \|A\|\|A^{-1}\| \tag{11}$$

or

$$\kappa = \alpha\|A\|\|A^{-1}\| \quad \text{where } \alpha = \frac{\|x\|}{\|Ax\|}\bigg/\|A^{-1}\| \tag{12}$$

Condition of Matrix-Vector Multiplication

Theorem

Let $A \in \mathbb{C}^{m \times m}$ *be nonsingular and consider the equation* $Ax = b$*. The problem of computing* b*, given* x*, has condition number*

$$\kappa = \|A\| \frac{\|x\|}{\|b\|} \leq \|A\| \|A^{-1}\| \tag{13}$$

with respect ro perturbation of x*. The problem of computing* x*, given* b*, has condition number*

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|x\|} \leq \|A\| \|A^{-1}\| \tag{14}$$

with respect to perturbation of b*. If* $\|\cdot\| = \|\cdot\|_2$*, then equality holds in* (13) *if* x *is a multiple of a right singular vector of* A *corresponding to the minimal singular vlaue* σ_m*, and equality holds in* (14) *if* b *is a multiple of a left singular vector of* A *corresponding to the maximal singular value* σ_1*.*

Condition Number of a Matrix

Condition number of $A \in \mathbf{C}^{m\times m}$, denoted by $\kappa(A)$:

$$\kappa(A) = \|A\|\|A^{-1}\| = \frac{\sigma_1}{\sigma_m} \tag{15}$$

For a rectangular matrix $A \in \mathbf{C}^{m\times n}$ of full rank, $m \geq n$, the condition number is defined in terms of the pseudoinverse: $\kappa(A) = \|A\|\|A^{+}\|$. So,

$$\kappa(A) = \frac{\sigma_1}{\sigma_n} \tag{16}$$

Condition of a System of Equations

In theorem, we held A fixed and perturbed x or b. What happens if we perturb A? Specifically, ler us hold b fixed and consider the vehavior of the problem $A \mapsto x = A^{-1}b$ when A is perturbed by δA. Then x must change bt δx, where

$$\begin{aligned}(A+\delta A)(x+\delta x) = b &\Rightarrow Ax + A(\delta x) + (\delta A)x + (\delta A)(\delta x) = b \\ &\Rightarrow A(\delta x) + (\delta A)x + (\delta A)(\delta x) = 0 \\ &\Rightarrow A(\delta x) + (\delta A)x = 0 \\ &\Rightarrow \delta x = -A^{-1}(\delta A)x \\ &\Rightarrow \|\delta x\| \leq \|A^{-1}\|\|\delta A\|\|x\|\end{aligned}$$

or equivalently,

$$\frac{\|\delta x\|}{\|x\|}\bigg/\frac{\|\delta A\|}{\|A\|} \leq \|A^{-1}\|\|A\| = \kappa(A).$$

Condition of a System of Equations

Theorem

Let b be fixed and consider the problem of computing $x = A^{-1}b$, where A is square and nonsingular. The condition number of this problem with respect ro perturbations in A is

$$\kappa = \|A\|\|A^{-1}\| = \kappa(A). \tag{17}$$

Two thorems in this lecture are of fundamental importance in numerical algebra, for ther determine how accurately one can solve systems of equations. If a problem $Ax = b$ contains an **ill-conditioned** matrix A, one must always expect to “loss $\log_{10} \kappa(A)$ digits” in computing the solution, except under very special circumstances.

Exercise

Exercise 12.1

Suppose A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.

Solution

Let $\sigma_1 \geq \cdots \geq \sigma_{202}$ be the singular values of A. We know that $\sigma_1 = \|A\|_2 = 100$. If $\sigma_{202} > 1$, then $\sigma_i > 1$ for all i, and so

$$\|A\|_F = \sqrt{\sum_{i=1}^{202} \sigma_i^2} = \sqrt{\sigma_1^2 + \sum_{i=2}^{202} \sigma_i^2} > \sqrt{100^2 + 201} = \sqrt{10201} = 101.$$

Since $\|A\|_F = 101$, it cannot be the case that $\sigma_{202} > 1$: $\sigma_{202} \leq 1$.

Exercise

Exercise 12.1

Suppose A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.

Solution

Furthermore, this upper bound is attainable. If $\sigma_2 = \cdots = \sigma_{202} = 1$, then

$$\|A\|_F = \sqrt{\sum_{i=1}^{202} \sigma_i^2} = \sqrt{100^2 + 201} = 101.$$

This leads to the sharpest possible lower bound on $\kappa(A)$, which is given by

$$\kappa(A) = \frac{\sigma_1}{\sigma_{202}} \geq \frac{100}{1} = 100.$$

Thank you!