Mathematics, Pusan National University

Introduction to Zhang Neural Network And Solving Time-varying Matrix Equations

Junior Math Colloquium

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Content

About Me

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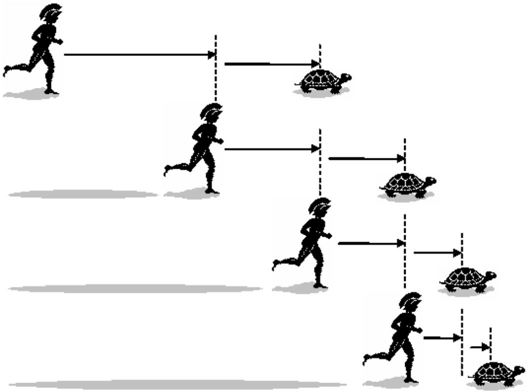

Figure 1: Zeno's paradoxes

In **mathematics**, Zeno's paradoxes is **false**.

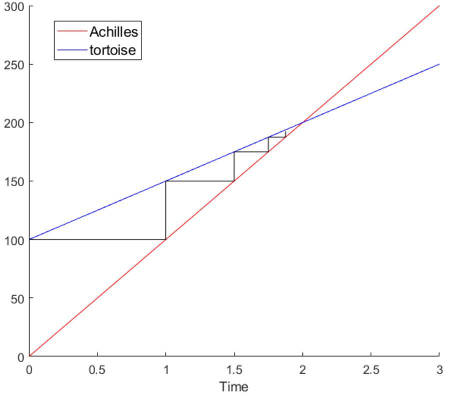

Figure 2: Obviously, Achilles can overtakes the tortoise!

But in **computer science**, Zeno’s paradox is **TRUE**!

Lagging error phenomenon

Consider the time-varying reciprocal problem in the following form:

$$f(x(t), t) = a(t)x(t) - 1 = 0 \in \mathbb{R}, t \in [0, -\infty) \quad (1)$$

where $a(t) \neq 0 \in \mathbb{R}$ denotes a smoothly time-varying scalar with $\dot{a}(t) \in \mathbb{R}$ denoting the time derivative of $a(t)$.

aim : Finding the $x(t) \in \mathbb{R}$ to make (1) hold true at any time $t \in [0, -\infty)$.
And denote $x^*(t)$ as the theoretical time-varying reciprocal of $a(t)$, i.e., mathematically, $x^*(t) = 1/a(t)$ in (1).

Lagging error phenomenon

Remark

This $x^*(t)$ is given symbolically for better understanding and solution comparison, whose the computation of $1/a(t)$ at every single time instant t is less practical in real-life applications. When we compute $1/a(t)$ at a time instant t, as the computation consumes time Δt inevitably, the value of $a(t)$ is changing during the computation procedure. This is the so-called **lagging error phenomenon**.

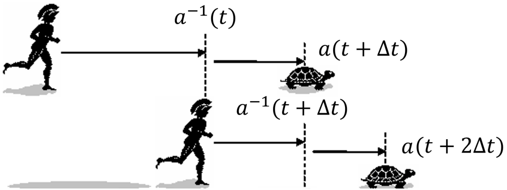

Figure 3: Achilles never can overtakes the tortoise... in computer!

Application

- Control theory : Real-time tracking
  - GPS

  - Robot arm

Zhang dynamics (ZD) has been formally proposed by Zhang et al. for various time-varying problems solving.

Concept of Zhang dynamics

Zhang dynamics(ZD) is a special type of neural dynamics that has been formally proposed by Zhang et al. for various time-varying problems solving.

According to Zhang et al.'s neural-dynamics design method, the ZD is designed based on an indefinite Zhang function (ZF) as the error-monitoring function.

To lay a basis for further discussion, the design procedure for a ZD model is presented as follows.

1. Define an indefinite ZF as the error-monitoring function to monitor the process of time-varying reciprocal finding.
2. To force $e(t)$ globally and exponentially converge to zero, we choose its time derivative $\dot{e}(t)$ via the following ZD design formula,

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma e(t), \qquad (2)$$

   where design parameter $\gamma > 0 \in \mathbb{R}$.
3. By expanding the ZD design formula (2), the dynamic equation of a ZD model is thus established for time-varying reciprocal finding.

Theorem 1.1

As for the ZD design formula (2) which is also a dynamic system, starting from an initial error $e(0) \in \mathbb{R}$, the error function $e(t) \in \mathbb{R}$ globally and exponentially converges to zero with rate γ.

Proof.

For (2), by calculus, we obtain its analytical solution as $e(t) = e(0)exp(-\gamma t)$. Based on the definition of global and exponential convergence, we can draw the conclusion that, starting from any $e(0)$, $e(t)$ globally and exponentially converges to zero with rate γ, as time t tends to infinity. □

Concept of Zhang Dynamics & Zhang Function

$$f(x(t),t) = a(t)x(t) - 1 = 0 \in \mathbb{R}, t \in [0, -\infty)$$

For real-time solution of time-varying reciprocal problem (1), we define the following four different ZFs:

$$e(t) = x(t) - \frac{1}{a(t)}, \tag{3}$$

$$e(t) = a(t) - \frac{1}{x(t)}, \tag{4}$$

$$e(t) = a(t)x(t) - 1, \tag{5}$$

$$e(t) = \frac{1}{a(t)x(t)} - 1. \tag{6}$$

$$\dot{e}(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma e(t)$$

Example of ZD model

Let us consider the ZD design formula (2) and ZF (3). Then, we have

$$\dot{x}(t) + \frac{1}{a^2(t)}\dot{a}(t) = -\gamma\left(x(t) - \frac{1}{a(t)}\right),$$

which is rewritten as

$$a^2(t)\dot{x}(t) = -\dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right). \tag{7}$$

Thus, we obtain ZD model (7) for time-varying reciprocal finding.

For ZD model (7),

$$\dot{x}(t) = \left(1 - a^2(t)\right)\dot{x}(t) - \dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right).$$

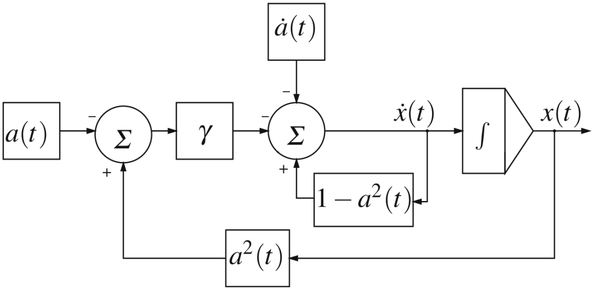

Figure 4: Block diagrams of ZD models (7) for time-varying reciprocal finding

Similarly, we obtain ZD models using ZFs equations (4)–(6), respectively.

ZF	ZD model
(3)	$a^2(t)\dot{x}(t) = -\dot{a}(t) - \gamma\left(a^2(t)x(t) - a(t)\right)$
(4)	$\dot{x}(t) = -\dot{a}(t)x^2(t) - \gamma\left(a(t)x^2(t) - x(t)\right)$
(5)	$a(t)\dot{x}(t) = -\dot{a}(t)x(t) - \gamma(a(t)x(t) - 1)$
(6)	$a(t)\dot{x}(t) = -\dot{a}(t)x(t) + \gamma\left(a(t)x(t) - a^2(t)x^2(t)\right)$

Table 1: Different ZFs resulting in different ZD models for time-varying reciprocal finding

Following proposition shows the convergence properties of the proposed ZD model (7) for time-varying reciprocal finding.

Proposition

Consider a smoothly time-varying scalar $a(t) \neq 0 \in \mathbb{R}$ involved in time-varying reciprocal problem (1). Starting from randomly-generated initial state $x(0) \neq 0 \in \mathbb{R}$ which has the same sign as $a(0)$, the neural state $x(t)$ of ZD model (7) derived from ZF (3) exponentially converges to the theoretical time-varying reciprocal $x^*(t)$ of $a(t)$ [i.e., $a^{-1}(t)$].

We will prove

$$A(t)X(t) - I = 0 \in \mathbb{R}^{n\times n} \tag{8}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is the smoothly time-varying nonsingular coefficient matrix. Note that $A(t)$ together with its time derivative $\dot{A}(t) \in \mathbb{R}^{n\times n}$ is assumed to be known or measurable. Generally, if the time-varying matrix $A(t) \in \mathbb{R}^{m\times n}$ is of full-rank, i.e., $\mathrm{rank}(A) = \min\{m, n\}$ at any time instant $t \in [0, +\infty)$, then the unique time-varying pseudoinverse/inverse $A^{+}(t)$ for matrix $A(t)$

$$A^{+}(t) = \begin{cases} \left(A^{\mathrm{T}}(t)A(t)\right)^{-1} A^{\mathrm{T}}(t), & \text{if } m > n \\ A^{-1}(t), & \text{if } m = n \\ A^{\mathrm{T}}(t)\left(A(t)A^{\mathrm{T}}(t)\right)^{-1}, & \text{if } m < n \end{cases} \tag{9}$$

Time-Varying Matrix Inversion

ZD design formula (2) is further generalized as follows

$$\dot{E}(t) = \frac{dE(t)}{dt} = -\gamma E(t), \tag{10}$$

where design parameter $\gamma \in \mathbb{R}$ is defined the same as before.

Specifically, for solving time-varying matrix-inversion problem (8), we define different ZFs as below:

$$E(t) = X(t) - A^{-1}(t) \tag{11}$$

$$E(t) = A(t) - X^{-1}(t) \tag{12}$$

$$E(t) = A(t)X(t) - I, \tag{13}$$

$$E(t) = X(t)A(t) - I, \tag{14}$$

$$E(t) = (A(t)X(t))^{-1} - I, \tag{15}$$

$$E(t) = (X(t)A(t))^{-1} - I. \tag{16}$$

Before constructing different ZD models from different ZFs, we present the following theorem for further discussion.

Time-Varying Matrix Inversion

Theorem

The time derivative of the time-varying matrix inverse $A^{-1}(t)$ is formulated as $\dot{A}^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$.

Proof.

Since $A(t)A^{-1}(t) = I \in \mathbb{R}^{n\times n}$, we have

$$\frac{\mathrm{d}\left(A(t)A^{-1}(t)\right)}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}t} = \mathbf{0} \in \mathbb{R}^{n\times n}.$$

Expanding the above equation, we obtain

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}A^{-1}(t) + A(t)\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = \mathbf{0} \in \mathbb{R}^{n\times n},$$

which is further rewritten as

$$A(t)\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -\frac{\mathrm{d}A(t)}{\mathrm{d}t}A^{-1}(t) = -\dot{A}(t)A^{-1}(t).$$

Proof.

Then, we have

$$\dot{A}^{-1}(t) = \frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

i.e.,

$$\dot{A}^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

□

Therefore, we have following fact:

$$\frac{\mathrm{d}X^{-1}(t)}{\mathrm{d}t} = -X^{-1}(t)\dot{X}(t)X^{-1}(t) \tag{17}$$

$$\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -A^{-1}(t)\dot{A}(t)A^{-1}(t) \tag{18}$$

$$\frac{\mathrm{d}(A(t)X(t))^{-1}}{\mathrm{d}t} = -(A(t)X(t))^{-1}\frac{\mathrm{d}(A(t)X(t))}{\mathrm{d}t}(A(t)X(t))^{-1} \tag{19}$$

Time-Varying Matrix Inversion

$$E(t) = X(t) - A^{-1}(t), \quad \dot{E}(t) = \frac{dE(t)}{dt} = -\gamma E(t)$$

Considering ZD design formula (10), ZF (11), and equation (18), we have

$$A(t)\dot{X}(t)A(t) = -\gamma(A(t)X(t) - I)A(t) - \dot{A}(t), \tag{20}$$

which is also rewritten in the following explicit form:

$$\dot{X}(t) = \dot{X}(t) + (A(t)\dot{X}(t) - \gamma(A(t)X(t) - I))A(t) - \dot{A}(t)$$

Therefore, based on ZF (11), we obtain ZD model (20) for time-varying matrix inversion.

Time-Varying Matrix Inversion

Similarly, we obtain ZD models using ZFs equations (11)–(16), respectively.

ZF	ZD model
(11)	$\dot{X}(t) = \dot{X}(t) + (A(t)\dot{X}(t) - \gamma(A(t)X(t) - I))A(t) - \dot{A}(t)$
(12)	$\dot{X}(t) = -X^{-1}(t)\dot{X}(t)X^{-1}(t) - \gamma X(t)(A(t)X(t) - I)$
(13)	$\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)$
(14)	$\dot{X}(t) = \dot{X}(t)(I - A(t)) - X(t)\dot{A}(t) - \gamma(X(t)A(t) - I)$
(15)	$\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)A(t)X(t)$
(16)	$\dot{X}(t) = \dot{X}(t)(I - A(t)) - X(t)\dot{A}(t) - \gamma X(t)A(t)(X(t)A(t) - I)$

Table 2: Different ZFs resulting in different ZD models (depicted in explicit dynamics for modeling purposes) for time-varying matrix inversion

Theorem

Let us consider a smoothly time-varying nonsingular matrix $A(t) \in \mathbb{R}^{n\times n}$ in (8). Starting from an initial state $X(0) \in \mathbb{R}^{n\times n}$, the state matrix $X(t)$ of ZD model (20) derived from ZF (11) globally and exponentially converges to the theoretical time-varying inverse $A^{-1}(t)$ of matrix $A(t)$.

$$\dot{X}(t) = \dot{X}(t) + (A(t)\dot{X}(t) - \gamma(A(t)X(t) - I))A(t) - \dot{A}(t)$$

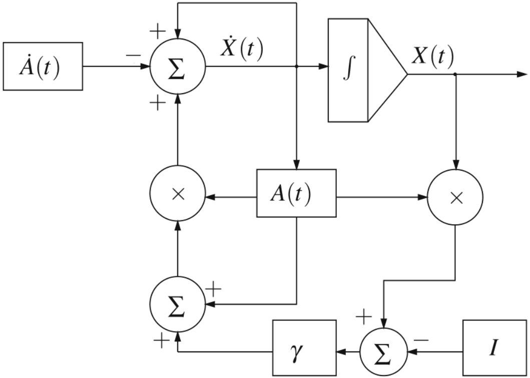

Figure 5: Block diagrams of ZD model (11) for time-varying matrix inversion

Time-Varying Matrix Inversion

Let us consider the time-varying matrix-inversion problem with the following time-varying matrix $A(t)$.

$$A(t) = \begin{bmatrix} \sin(5t) & \cos(5t) \\ -\cos(5t) & \sin(5t) \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad (21)$$

By algebraic operations, the theoretical time-varying inverse of $A(t)$ is given as

$$X^*(t) = A^{-1}(t) = \begin{bmatrix} \sin(5t) & -\cos(5t) \\ \cos(5t) & \sin(5t) \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad (22)$$

Thus, we can use such a theoretical solution to compare with the solutions of corresponding ZD models and then check the correctness of the models' solutions.

$$E(t) = A(t)X(t) - I$$
$$\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)$$

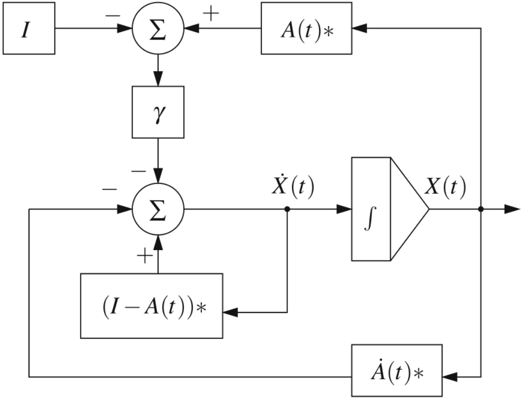

Figure 6: Block diagrams of ZD model using ZF (15) for time-varying matrix inversion

$$\dot{X}(t) = (I - A(t))\dot{X}(t) - \dot{A}(t)X(t) - \gamma(A(t)X(t) - I)$$

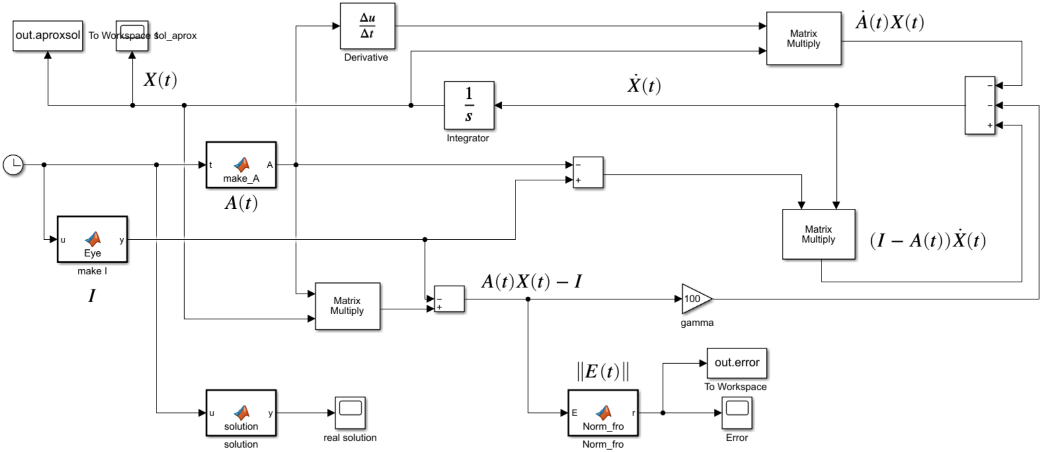

Figure 7: Overall Simulink modeling of ZD model using ZF (13) for time-varying matrix inversion

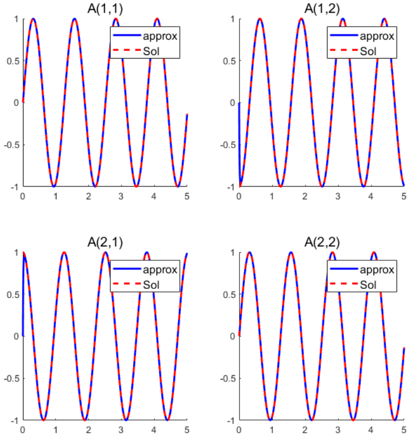

Figure 8: Result of ZNN to get inverse of time-varying matrix

Time-varying quadratic matrix equation

Consider a time-varying quadratic matrix equation

$$\mathcal{F}(t) = A(t)(X(t))^2 + B(t)X(t) + C(t) = 0 \tag{23}$$

where $A(t), B(t), C(t) \in \mathbb{R}^{n\times n}$ are given and $X(t) \in \mathbb{R}^{n\times n}$ is unknown matrix.

We will compare three methods for solving (9). These are Fixed point iteration(FPI), Newton's method(NM), and ZNN.

In this experiments, we set $A(t), B(t), C(t)$ as following:

$$A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix},$$

$$C(t) = \begin{bmatrix} \cos(t)^2 - 2\cos(t)\sin(t) - \sin(t)^2 & \sin(t)^2 - \cos(t)^2 - 2\cos(t)\sin(t) \\ 2\cos(t)\sin(t) + \cos(t)^2 - \sin(t)^2 & \cos(t)^2 - 2\cos(t)\sin(t) - \sin(t)^2 \end{bmatrix}.$$

Then the solution matrix is $S(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{bmatrix}$.

We use the following error function for each method:

for fixed time t,

$$Error(t) = \|S(t_{cal}) - X(t)\|_F$$

where $t_{cal} = t$ + calculation time of each method.

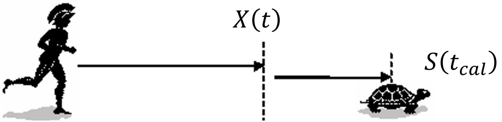

Figure 9: Time passes even while the algorithm is running.

For fixed time t, find $X(t)$ for fixed $A(t), B(t), C(t)$ using Newton's method.

Algorithm 1: Newton's method(NM)

Input: $A(t), B(t), C(t)$, tolerence: tol
Output: solution: X, calculation time: t_{cal}

$X \leftarrow \text{zeros}(2,2)$ `// Starting NM with zero initial matrix.`

tic `// Calculate start. Time is still running.`

while $res > tol$ **do**

$$\text{vec}\, H = -(I \otimes (AX + B) + X^{\top} \otimes A)^{-1}\, \text{vec}(AX^2 + BX + C)$$

$X_{new} \leftarrow X + H$

$res \leftarrow \|X_{new} - X\|_F$

$X \leftarrow X_{new}$

end

$t_{cal} \leftarrow t + toc$ `// Calculation end.`

Numerical Experiments

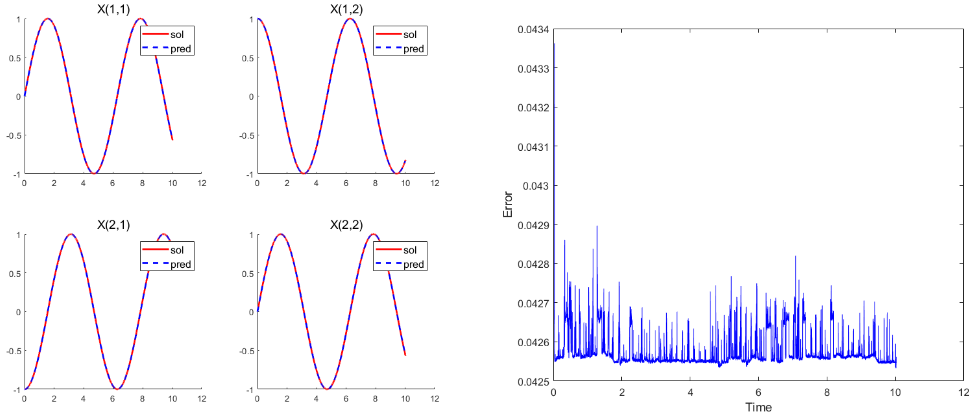

Figure 10: Result of Newton's method

For fixed time t, find $X(t)$ for fixed $A(t), B(t), C(t)$ using Fixed point iteration.

Algorithm 2: Fixed point iteration(FPI)

Input: $A(t), B(t), C(t)$, tolerence: tol
Output: solution: X, calculation time: t_{cal}

$X \leftarrow \mathrm{zeros}(2,2)$ `// Starting FPI with zero initial matrix.`
tic `// Calculate start. Time is still running.`
while $res > tol$ **do**
  $X_{new} \leftarrow (-B - AX)^{-1}C$
  $res \leftarrow \|X_{new} - X\|_F$
  $X \leftarrow X_{new}$
end
$t_{cal} \leftarrow t + toc$ `// Calculation end.`

Numerical Experiments

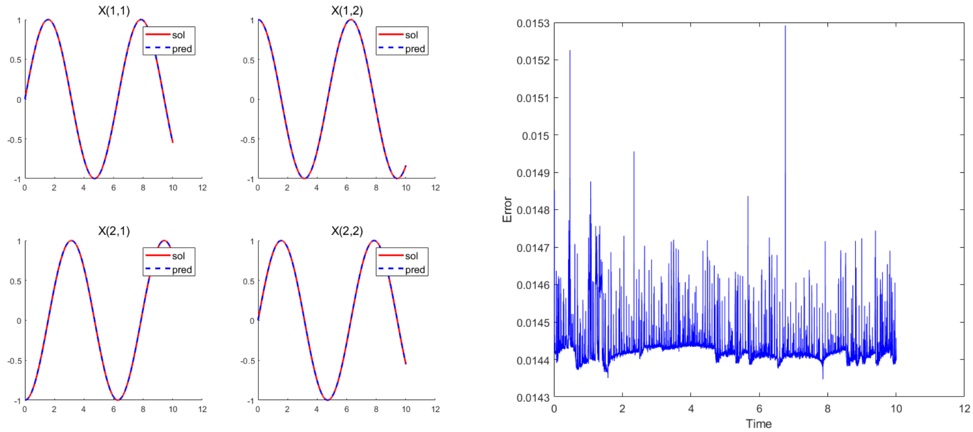

Figure 11: Result of Fixed point iteration

Let ZF as below:

$$E(t) = A(t)(X(t))^2 + B(t)X(t) + C(t) \tag{24}$$

And considering ZD design formula (2)

$$\begin{aligned}\dot{E}(t) &= \frac{dE(t)}{dt} \\ &= \dot{A}(t)(X(t))^2 + A\dot{X}(t)X(t) + AX(t)\dot{X}(t) + \dot{B}(t)X(t) + B(t)\dot{X}(t) + \dot{C}(t) \\ &= -\gamma(A(t)(X(t))^2 + B(t)X(t) + C(t))\end{aligned}$$

Then, we can obtain ZD model using ZF equation,

$$\begin{aligned}\dot{X}(t) =&(I - A(t)X(t) - B(t))\dot{X}(t) - A(t)\dot{X}(t)X(t) - \dot{A}(t)(X(t))^2 \\ &- \dot{B}(t)X(t) - \gamma(A(t)(X(t))^2 + B(t)X(t) + C(t))\end{aligned}$$

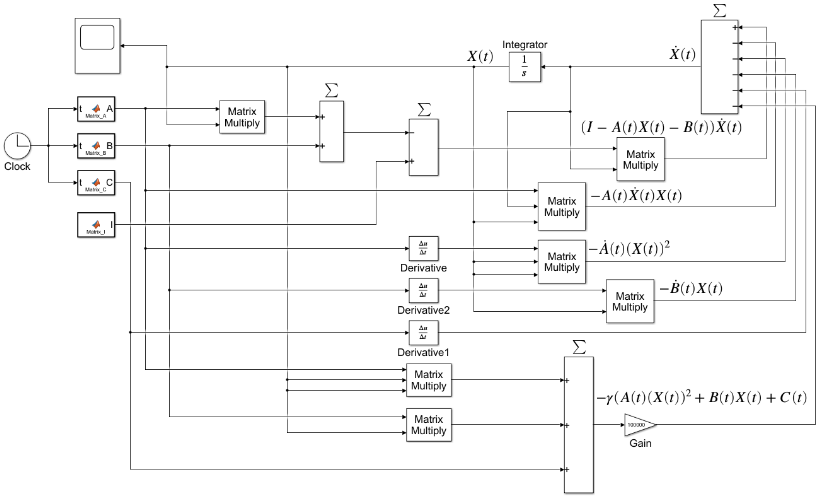

Figure 12: ZNN Simulink Model for Solving QME

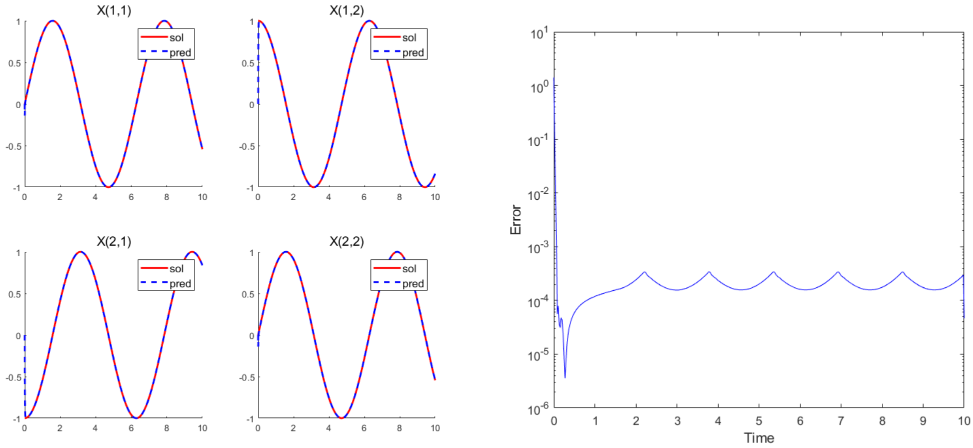

Figure 13: Result of Zhang Neural Network

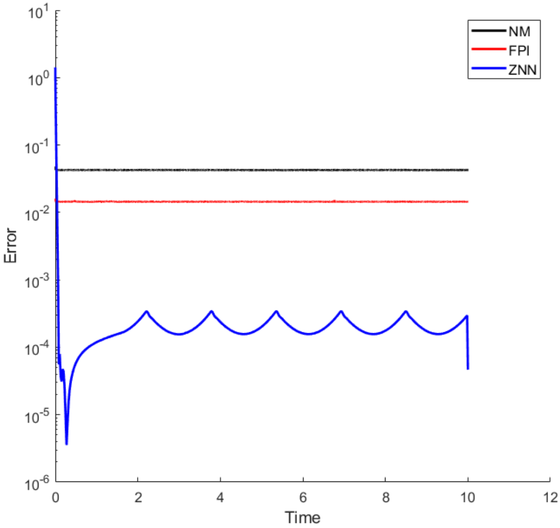

Figure 14: Error comparison for each method

Summary

- Understanding the time-varying problem
- Introduction to Zhang dynamic and Zhang function to create Zhang Neural Network
- Solve the time-varying matrix equation using ZNN
- Check the advantages of ZNN by comparing with other methods

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Thank you!