Mathematics, Pusan National University

Tensor Analysis

1.2 Tensor Multiplications

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Content

Definition (Tensor Outer Product)

We use $\otimes$ to denote tensor outer product; that is for any two tensors $\mathscr{A} \in T_{m,n}$ and $\mathscr{B} \in T_{p,n}$,

$$\mathscr{A} \otimes \mathscr{B} = \left(a_{i_1 \dots i_m} b_{i_{m+1} \dots i_{m+n}}\right) \tag{1}$$

symmetric rank-one tensor

$$\mathbf{x}^{\otimes k} \equiv \underbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{k \text{ times}} = \left(x_{i_1} \cdots i_k\right) \in T_{k,n} \tag{2}$$

Obviously, $\mathbf{x}^{\otimes k} \in S_{k,n}$, and it called a symmetric rank-one tensor when $\mathbf{x} \neq \mathbf{0}$.

rank-one tensor

More generally, let $\mathbf{x}^{(i)} = \left(x_1^{(i)}, \ldots, x_n^{(i)}\right)^T \in \mathbb{R}^n$ for $i \in [m]$ and $\alpha \in \mathbb{R}$. Then $\alpha\mathbf{x}^{(1)} \otimes \mathbf{x}^{(2)} \otimes \cdots \otimes \mathbf{x}^{(m)}$ is a tensor in $T_{m,n}$ with isd $(i_1, \ldots, i_m)$th entry as $\alpha x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$. Such a tensor (not necessarily symmetric) is called a rank-one tensor in $T_{m,n}$.

Definition (k-Mode Product)

For any $\mathscr{A} \in T_{m,n}$ and any $P = (p_{ij}) \in \mathbb{R}^{p\times n}$, and for any given $k \in [m]$, the k-mode product of $\mathscr{A}$ and P, denoted as $\mathscr{A} \times_k P$, is defined by

$$(\mathscr{A} \times_k P)_{i_1\cdots i_{k-1} j i_{k+1}\cdots i_m} = \sum_{i_k=1}^{n} a_{i_1\cdots i_{k-1} i_k i_{k+1}\cdots i_m} p_{i,i_k}, \quad \forall i_l \in [n], l \in [m], l \neq k, \forall j \in [p] \tag{3}$$

By this product, the size of tensor is changed from $n \times \cdots \times n$ to $n \times \cdots \times p \times \cdots \times n$.

linear operator $P^m(\cdot)$

If we do the k-mode product of $\mathscr{A}$ and P for all possible $k \in [n]$ as

$$P^m(\mathscr{A}) = \mathscr{A} \times_1 P \times_2 \cdots \times_m P$$

More specifically,

$$P^m(\mathscr{A}) = \left(\sum_{i_1,\ldots,i_m=1}^{n} a_{i_1,\ldots,i_m} p_{j_1 i_1} \cdots p_{j_m i_m} \right) \in T_{m,p}, \quad \forall \mathscr{A} = \left(a_{i_1,\ldots,i_m}\right) \in T_{m,n}. \tag{4}$$

For $\mathbf{x}^T = (x_1, \dots, x_n)$, the following frequently used notations are given as below:

$$\mathscr{A}\mathbf{x}^{m-2} \equiv \mathscr{A} \times_3 \mathbf{x}^T \times_4 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_3, \dots, i_m = 1}^{n} a_{iji_3 \cdots i_m} x_{i_3} \cdots x_{i_m} \right) \in \mathbb{R}^{n \times n} \tag{5}$$

$$\mathscr{A}\mathbf{x}^{m-1} \equiv \mathscr{A} \times_2 \mathbf{x}^T \times_3 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_2, \dots, i_m = 1}^{n} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right) \in \mathbb{R}^{n} \tag{6}$$

$$\mathscr{A}\mathbf{x}^{m} \equiv \mathscr{A} \times_1 \mathbf{x}^T \times_2 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_1, \dots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m} \right) \in \mathbb{R} \tag{7}$$

Definition (Inner Product)

For any two tensor $\mathscr{A} = \left(a_{i_1 \cdots i_m}\right), \mathscr{B} = \left(b_{i_1 \cdots i_m}\right) \in T_{m,n}$, the inner product of $\mathscr{A}$ and $\mathscr{B}$, denoted as $\mathscr{A} \bullet \mathscr{B}$, is defined as

$$\mathscr{A} \bullet \mathscr{B} = \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}. \tag{8}$$

Frobenious norm of $\mathscr{A}$

$$\|\mathscr{A}\|_F = \sqrt{\mathscr{A} \bullet \mathscr{A}}$$

Definition (Hadamard Product)

For any two tensor $\mathscr{A} = \left(a_{i_1 \ldots i_m}\right), \mathscr{B} = \left(b_{i_1 \ldots i_m}\right) \in T_{m,n}$, the Hadamard product of $\mathscr{A}$ and $\mathscr{B}$, denoted as $\mathscr{A} \circ \mathscr{B}$, is defined as

$$\mathscr{A} \circ \mathscr{B} = \left(a_{i_1 \ldots i_m} b_{i_1 \ldots i_m}\right) \in T_{m,n} \tag{9}$$

Next : 1.3 Tensor Decomposition and Tensor Rank