

Mathematics, Pusan National University

Tensor Analysis

1.2 Tensor Multiplications

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Tensor Multiplications

- Tensor Outer Product

- k -Mode Product

- Inner Product

- Hadamard Product



Definition (Tensor Outer Product)

We use \otimes to denote tensor outer product; that is for any two tensors $\mathcal{A} \in T_{m,n}$ and $\mathcal{B} \in T_{p,n}$,

$$\mathcal{A} \otimes \mathcal{B} = (a_{i_1 \dots i_m} b_{i_{m+1} \dots i_{m+n}}) \quad (1)$$



symmetric rank-one tensor

$$\mathbf{x}^{\otimes k} \equiv \underbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{k \text{ times}} = (x_{i_1} \cdots i_k) \in T_{k,n} \quad (2)$$

Obviously, $\mathbf{x}^{\otimes k} \in S_{k,n}$, and it called a symmetric rank-one tensor when $\mathbf{x} \neq \mathbf{0}$.

rank-one tensor

More generally, let $\mathbf{x}^{(i)} = (x_1^{(i)}, \dots, x_n^{(i)})^T \in \mathbb{R}^n$ for $i \in [m]$ and $\alpha \in \mathbb{R}$.

Then $\alpha \mathbf{x}^{(1)} \otimes \mathbf{x}^{(2)} \otimes \cdots \otimes \mathbf{x}^{(m)}$ is a tensor in $T_{m,n}$ with $\text{isd}(i_1, \dots, i_m)$ th entry as $\alpha x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$. Such a tensor (not necessarily symmetric) is called a rank-one tensor in $T_{m,n}$.



Definition (k -Mode Product)

For any $\mathcal{A} \in T_{m,n}$ and any $P = (p_{ij}) \in \mathbb{R}^{p \times n}$, and for any given $k \in [m]$, the k -mode product of \mathcal{A} and P , denoted as $\mathcal{A} \times_k P$, is defined by

$$(\mathcal{A} \times_k P)_{i_1 \dots i_{k-1} j i_{k+1} \dots i_m} = \sum_{i_k=1}^n a_{i_1 \dots i_{k-1} i_k i_{k+1} \dots i_m} p_{i, i_k}, \quad (3)$$
$$\forall i_l \in [n], l \in [m], l \neq k, \forall j \in [p]$$

By this product, the size of tensor is changed from $n \times \dots \times n$ to $n \times \dots \times p \times \dots \times n$.



linear operator $P^m(\cdot)$

If we do the k -mode product of \mathcal{A} and P for all possible $k \in [n]$ as

$$P^m(\mathcal{A}) = \mathcal{A} \times_1 P \times_2 \cdots \times_m P$$

More specifically,

$$P^m(\mathcal{A}) = \left(\sum_{i_1, \dots, i_m=1}^n a_{i_1, \dots, i_m} p_{j_1 i_1} \cdots p_{j_m i_m} \right) \in T_{m,p}, \quad (4)$$
$$\forall \mathcal{A} = (a_{i_1, \dots, i_m}) \in T_{m,n}.$$

Tensor Multiplications

k -Mode Product



For $\mathbf{x}^T = (x_1, \dots, x_n)$, the following frequently used notations are given as below:

$$\mathcal{A} \mathbf{x}^{m-2} \equiv \mathcal{A} \times_3 \mathbf{x}^T \times_4 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_3, \dots, i_m=1}^n a_{ij i_3 \dots i_m} x_{i_3} \cdots x_{i_m} \right) \in \mathbb{R}^{n \times n} \quad (5)$$

$$\mathcal{A} \mathbf{x}^{m-1} \equiv \mathcal{A} \times_2 \mathbf{x}^T \times_3 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right) \in \mathbb{R}^n \quad (6)$$

$$\mathcal{A} \mathbf{x}^m \equiv \mathcal{A} \times_1 \mathbf{x}^T \times_2 \cdots \times_m \mathbf{x}^T = \left(\sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \cdots x_{i_m} \right) \in \mathbb{R} \quad (7)$$



Definition (Inner Product)

For any two tensor $\mathcal{A} = (a_{i_1 \dots i_m})$, $\mathcal{B} = (b_{i_1 \dots i_m}) \in T_{m,n}$, the inner product of \mathcal{A} and \mathcal{B} , denoted as $\mathcal{A} \bullet \mathcal{B}$, is defined as

$$\mathcal{A} \bullet \mathcal{B} = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} b_{i_1 \dots i_m}. \quad (8)$$

Frobenious norm of \mathcal{A}

$$\|\mathcal{A}\|_F = \sqrt{\mathcal{A} \bullet \mathcal{A}}$$



Definition (Hadamard Product)

For any two tensor $\mathcal{A} = (a_{i_1 \dots i_m})$, $\mathcal{B} = (b_{i_1 \dots i_m}) \in T_{m,n}$, the Hadamard product of \mathcal{A} and \mathcal{B} , denoted as $\mathcal{A} \circ \mathcal{B}$, is defined as

$$\mathcal{A} \circ \mathcal{B} = (a_{i_1 \dots i_m} b_{i_1 \dots i_m}) \in T_{m,n} \quad (9)$$

An abstract graphic consisting of several flowing, curved lines in shades of light blue and white, resembling a stylized wave or a dynamic motion. The lines are layered and have a soft, ethereal quality, with some lines appearing to have a slight glow or particle effect.

Next : 1.3 Tensor Decomposition and Tensor Rank